ELEMENTARY GEOMETRY

ACCORDING TO
MODERN METHOD

BY

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PREFACE.

The following pages contain all the important propositions which are given in books on Elementary Geometry according to the modern method, as distinguished from the method of Euclid, and a knowledge of which is conducive to general culture, and necessary for further mathematical study.

The points of difference between the modern method and the method of Euclid may be shortly stated thus :—

i. Euclid, from a strict regard for regularity, does not allow any construction though required only for demonstrating a proposition, to be assumed as effected, without shewing previously how it is to be made ; whereas, according to the modern method, for convenience of treatment, simple constructions, such as the bisection of a straight line or an angle, or the drawing of a straight line parallel or perpendicular to another, are assumed to be effected, when necessary merely for the purpose of proving a theorem.

ii. Euclid, with a view to dispense, as far as possible, with the aid of instruments in effecting his constructions, grants only three things to be done, namely, the drawing of a straight line from one point to another, the production of a given straight line indefinitely, and the describing of a circle from any centre at any distance from that centre ; that is, he allows the use of only two instruments, an ungraduated straight ruler for drawing straight lines, but not for taking any measurements, and a pair of compasses for describing a circle, but not for transferring, or for taking the measure of, any length from one position to another in any other way ; whereas, in the modern method, these things are allowed to be effected with the aid of instruments.

iii. Euclid deals with magnitudes directly, without resorting to their numerical equivalents in terms of any units of measurement, and is thus able to treat commensurable and incommensurable magnitudes alike, but at the same time, he is obliged to have recourse to a cumbrous and by no means obvious criterion of proportionality; whereas in the modern method, magnitudes, especially when considered with reference to proportionality, are taken to be represented by their numerical equivalents, the difficulty in the case of incommensurables being got rid of by the fact that they may be expressed numerically to any required degree of accuracy by adopting adequately small units.

iv. Euclid is content with merely demonstrating his propositions, without helping the learner much to see why a particular construction is made, or a particular series of steps in a demonstration is taken; and his propositions stand detached, without there being any attempt at generalisation of allied truths; while Modern Geometry seeks to indicate to the learner, and to help him in finding out for himself, the reasons for the processes of construction and demonstration adopted, and to present generalised statements and proofs of connected propositions.

v. Besides the above mentioned general points of difference, there are some particular points, such as those connected with the definitions of an angle, a diameter of a circle, and a tangent to a circle. Euclid's definition of an angle will exclude from consideration a re-entrant angle, and his definitions of a diameter and a tangent, though simple and sufficient for the circle, are inapplicable to many curves; whereas Modern Geometry, in order to make the definitions of those terms more comprehensive, introduces the idea of rotation of a line about a point into the definition of an angle, and regards the diameter as the locus of the middle points of a system of parallel chords, and the tangent as the limiting position of a secant.

It will thus be seen that Euclid's method, if it has the advantage of being more direct and more rigorous in form than the modern method, labours under the countervailing

disadvantage of being less comprehensive and more cumbrous; while the modern method, if it is less direct and less rigorous in form, has the compensating advantage of being more general and less burdensome, and better adapted to help progress in mathematical study.

It should be noticed that Euclid has the advantage of being well known and well suited for easy reference.

The advantages of Euclid's method at one time seemed to me to outweigh its disadvantages, and induced me to think that his Elements of Geometry, with suitable modifications, should be adopted as the text book in Geometry for the beginner. But it has since appeared to me to be necessary to lighten the labour of the student in acquiring a knowledge of Elementary Geometry, so that he may be able to spare time and energy for studying other subjects; and I am now of opinion that Euclid may well be replaced by Modern Geometry.

But if Euclid is to be superseded, our chief aim should be to help the beginner in the subject in learning, with each and within a short time, all the important elementary truths of Geometry. In this little book I have accordingly omitted all unimportant propositions, and tried to give the substance of the first six Books of Euclid in 50 Theorems and 25 Problems. In addition to the important propositions of Euclid, I have included a Problem (Problem 6 of Book III) for finding the numerical value of the ratio of the circumference of a circle to its diameter. This is taken with a slight modification from Legendre's Geometry, and is inserted here as a determination by simple elementary method, of the value of the important constant, π, which is usually left to be determined by the aid of the higher parts of Trigonometry, though the student has to assume the value as known at a much earlier stage.

To make the book complete, a few elementary propositions of Solid Geometry have also been included.

Problems have been separated from Theorems in each Book, and propositions have been sought to be arranged with due regard to order.

Symbols and abbreviations have been largely used, with a view to make the demonstrations not only shorter in appearance but more clear in reality. When once the student becomes familiar with the use of symbols, as he soon will, a demonstration presented in symbolical form will be grasped by him more readily as a whole, and followed more easily in its different steps, than a demonstration written out at length in ordinary language.

While the demonstrations given have been sought to be made clear, and explanatory notes have been given to elucidate important points, I have avoided encumbering the student with any superfluous help.

The Exercises given at the end of each Book are not many; but they will be found to be of sufficient variety. Moreover, the smallness of their number will, it is hoped, encourage students to attempt to solve them all, and to devote sufficient time and attention to each before giving it up, or asking for help. One Problem worked out by the student himself is worth more than a dozen solved with the help of others.

NARIKELDANGA
August, 1906.

G. D. BANERJEE.

PREFACE TO THE FOURTH EDITION.

In this edition, the additions and alterations made in the second edition have been retained. Of these, Note 2 at page 6, Theorem 16 at page 129, and Problem 3 at page 141 may deserve mention.

A few more Exercises have been added.

Problems have, as in the first edition, been placed after Theorems in each Book, because they almost invariably require the aid of Theorems to explain the reasons for the constructions involved, whereas Theorems can hardly be said to require the aid of any Problem that follows.

This book will, it is hoped, satisfy the requirements of the syllabus in Geometry prescribed by the University of Calcutta for the Matriculation and Intermediate Examinations.

December 26, 1907. G. D. B.

PREFACE TO THE SIXTH EDITION.

In this edition a few additions have been made, of which the General Note after the Axioms at page 5, and the additional methods of proving Theorem 21 of Book I and of solving Problems 5 and 6 of the same Book, may deserve mention.

The grouping together of connected Propositions is expressly indicated by the insertion of appropriate headings before the different groups.

July 22, 1910. G. D. B.

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BOOK I.

STRAIGHT LINES, ANGLES, AND RECTILINEAL FIGURES.

SECTION I. DEFINITIONS, AXIOMS, AND POSTULATES.

I. Definitions.

1. **Geometry** is the science which treats of Solids, Surfaces, Angles, Lines and Points.

2. A **solid** is that which has length, breadth, and thickness.

3. A **surface** is that which has only length and breadth.

Note. The boundaries of a solid are surfaces.

4. A **line** is length without breadth.

Note. The boundaries of a surface are lines.

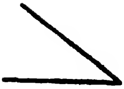
5. A **point** is that which has position but no magnitude.

Note. The extremities of a line are points.

6. A **straight line** is a line which has the same direction throughout its whole length.

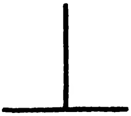
7. A **plane surface** or a **plane** is a surface in which any two points being taken, the straight line between them lies wholly in that surface.

8. If two straight lines, which are not in the same straight line, meet they are said to be *inclined to one another*, and the inclination between them is called a **plane rectilineal angle** or simply an **angle**.



9. If two straight lines, which are in the same plane, are such that being produced ever so far both ways, they do not meet, they are said to be **parallel** to one another.

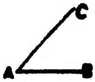
10. When one straight line stands on another so as to make the adjacent angles equal to one another, each of the angles is called a **right angle,** and each of the lines is said to be **perpendicular** to the other.



NOTE. The magnitude of an angle is estimated by the amount of *rotation* of one of the lines containing it about their point of intersection which is necessary to bring it to its actual position, supposing it to commence rotating after coincidence with the other line. Viewed in this way, an angle may be greater than two right angles; and such an angle is called a **re-entrant** angle.

A line is named by two letters placed at its extremities, and an angle by three letters whereof the middle one is placed at the point of intersection of the lines containing it, and the other two at their other extremities.

Thus the lines containing the angle in the annexed figure are named **AB**, **AC** and the angle between them, **BAC** or **CAB**.



When there is only one angle at a point as at **A**, it may be named as angle **A**. An unconnected line may be named by a single letter.

11. An angle less than a right angle is called an **acute angle.**

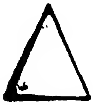
12. An angle greater than a right angle but less than two right angles, is called an **obtuse angle**.

13. A **rectilineal figure** is a figure bounded by straight lines.

It is called a **triangle,** if it is bounded by three straight lines, a **quadrilateral**, if bounded by four, and a **polygon**, if bounded by more than four straight lines.

14. An **equilateral triangle** is a triangle having three equal sides.

15. An **isosceles triangle** is a triangle having two equal sides.



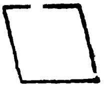
16. A **scalene triangle** is a triangle having three unequal sides.

17. A **parallelogram** is a four sided figure having its opposite sides parallel.

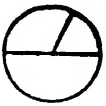
18. A **rectangle** is a parallelogram having a right angle.

19. A **square** is a rectangle having all its sides equal.

20. A **rhombus** is a parallelogram having all is sides equal.



21. A **circle** is a plane figure contained by one line which is called its **circumference**, and is such that all straight lines drawn from a certain fixed point within it to the circumference are equal to one another; and the fixed point within is called the **centre**.



22. A **radius** of a circle is a straight line drawn from the centre to the circumference.

23. A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

GENERAL NOTE. The foregoing definitions give the meanings of the terms defined, and imply that the things signified by those terms are possible things. Thus, it is implied that points, lines, parallel straight lines, and circles can be conceived to exist and can be supposed to be marked or drawn.

It is true that a line, however finely drawn, will have some breadth, and a point, however small the mark representing it, will have some magnitude ; but the breadth in the one case, and the magnitude in the other, are supposed not to exist, that is, are not taken into account. If this was not done, difficulties would arise. Thus, in bisecting a straight line, that is, in dividing it into two equal parts, the middle point, if we do not disregard its magnitude, will have to be divided into two equal parts, and the mark by which this division is effected will again have to be similarly divided, and so on, before the line can be said to be bisected.

24. An **axiom** is a self-evident truth.

25. A **postulate** is an assumption that a certain simple construction may be effected.

26. A **theorem** is a proposition stating a certain truth to be demonstrated.

27. A **problem** is a proposition proposing a certain construction to be effected.

28. When a proposition states that a certain *condition* being assumed, a certain *inference* follows, the condition assumed is called the **hypothesis**, and the inference following, the **conclusion** of the proposition

When two propositions are so related that the conclusion of the one is the hypothesis of the other, and the hypothesis of the former, the conclusion of the latter, the latter proposition is said to be the **converse** of the former.

II. Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be taken from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be taken from unequals, the remainders are unequal.

6. Things which are the same multiples of equals are equal.

7. Things which are the same parts of equals are equal.

8. The whole is greater than its part.

9. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

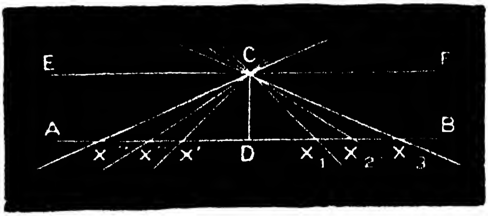
10. Two straight lines cannot enclose a space, nor can two straight lines have a common segment.

11. All right angles are equal.

12. Two intersecting straight lines cannot both be parallel to the same straight line.

NOTE. This is Playfair's axiom about parallel straight lines; and it is here adopted as being the simplest of all the axioms that have been proposed regarding parallels.

The following remarks may help the student in realizing the truth of this axiom, which asserts that through a given point there can be drawn one and only one straight line parallel to a given straight line.



Let AB be a given straight line, and C a given point without it. Let CD be the perpendicular from C on AB; and let a straight line rotate about the point C, starting from the position in which it is coincident with CD, and passing through the positions CX_1, CX_2, CX_3, ECF, CX″, CX″, CX′. The points of its intersection with AB, that is, X_1, X_2, X_3...on the right side of CD, move further and further from D until the rotating line comes to the position ECF, when the point of intersection moves to an infinite distance from D; and further rotation transfers the points of intersection to the left of CD, and makes those points, that is, X′, X″, X′ approach nearer and nearer to D. There is one position and only one, namely, ECF, in which the rotating line never meets AB; and it will be observed that this is the position in which the rotating line does not incline towards AB either on the right side or on the left side of CD.

GENERAL NOTE. 1. Axioms 1 to 8 apply to all sorts of measurable quantity, while axioms 9 to 12 refer to geometrical magnitudes only.

2. The converse of axiom 9 is not always true. Take for instance the case of a pair of shoes. See Book IV Theorem 16 Note 2.

3. Axiom 10 furnishes a test of the straightness of a line. Make an exact copy of the line, and place the line on its copy in different positions. If they coincide in *all possible* positions, the line is straight. Note that arcs of equal circles coincide in certain positions, but do not coincide when those positions are reversed. The straightness of a ruler may be

tested in the same way, by placing it side by side with another ruler and observing whether their edges coincide in every position or not.

4. Axiom 11 and definition 10 should be taken together.

Definition 10 furnishes a test of the correctness of a set square Draw a straight line on a plane such as a sheet of paper stretched on a board. Place one of the sides of the set square containing the right angle along the straigot line, and trace the other side on the paper. Reverse the square and observe whether the last mentioned side coincides with its tracing. If it does, the angle of the square is a right angle.

III. POSTULATES.

Let it be granted—

1. That a straight line may be drawn from any one point to any other point.

2. That a terminated straight line may be produced to any length in a straight line.

3. That a circle may be described with any point as centre and with any finite straight line as radius.

4. That a finite straight line may be bisected at a point.

5. That any angle may be bisected by a straight line.

6. That a straight line may be drawn perpendicular to a given straight line from any point in or without it.

7. That a straight line may be drawn parallel to a given straight line from any point without it.

8. That a straight line may be drawn from any point in a given straight line making a given angle with it.

NOTE 1. Postulates 1 and 2 assume the use of an ungraduated straight ruler, and postulate 3 assumes the use of a pair of compasses for describing a circle, and for transferring definite distances or lengths in a straight line.

Postulates 4 to 8 are assumed only for effecting certain simple constructions which may be necessary for the demonstration of theorems, and it is shown later (see Problems 2 to 6) how the constructions assumed as effected, can be made with the help of postulates 1 to 3, which are the only postulates that are really assumed.

NOTE 2. The constructions assumed in postulates 4 and 5 and the first part of postulate 6 may be supposed to be effected in the following manner, without the help of any instrument.

To bisect a straight line, suppose the plane in which it lies to be perfectly flexible and to be folded so that one of the extremities of the

line falls on the other; then the two parts into which the line is broken coincide and are equal, and the point at which it is folded is its middle point.

To bisect an angle, suppose the plane in which it lies to be perfectly flexible and to be folded so that one of the lines containing the angle falls on the other; then the two parts into which the angle is divided coincide and are equal, and the line about which the plane is folded, that is, the crease, is the bisector of the angle.

To draw a perpendicular to a straight line from a given point in it, produce the line if the given point is one of its extremities, and suppose the plane in which it lies to be perfectly flexible and to be folded so that the crease passes through the given point, and the two parts of the line on the two sides of that point fall on one another; then the angles formed by the crease with those two parts of the line coincide and are evidently right angles, and the crease is the perpendicular required.

The foregoing remarks point to some of the exercises in paper folding, in which the student should be practised, as a method of constructing geometrical figures.

SECTION II. THEOREMS.

Introductory Remarks. 1. The following symbols and abbreviations will be used in this book :—

pt.	*for*	point.
| *or* st. line	„	straight line.
∠	„	angle.
rt. ∠	„	right angle.
∥	„	parallel to, *or* is *or* are parallel to.
⊥	„	perpendicular to, *or* is *or* are perpendicular to.
△	„	triangle.
▱ *or* parm.	„	parallelogram.
▭ *or* rect.	„	rectangle.
□ *or* sq.	„	square.
⊙	„	circle.
○	„	circumference.
∵	„	because.
∴	„	therefore.
=	„	equal to *or* is *or* are equal to.
>	„	greater than *or* is *or* are greater than.
<	„	less than *or* is *or* are less than.
AB^2	„	square on AB.
AB.CD	„	rectangle contained by AB and CD.

In reading the book, and in stating orally the constructions and demonstrations in propositions, the student should use no abbreviated language, but should express himself in sentences that are complete and correct.

2. The student should try *to see the necessity* of every construction, and *to find out for himself* as far as he can, *the reason* for the different steps taken in the demonstration of a proposition.

3. The student should be careful not to assume the truth of any proposition except the axioms given above, and the theorems previously proved.

4. Though the student may *theoretically* assume the constructions mentioned in the postulates given above, as effected correctly, he should be careful *practically* to effect them correctly, and for that purpose to know how they can be made with the aid of instruments. For correct figures will not only help him in easily following demonstrations, but will often enable him readily to find out for himself the different steps of a demonstration.

The Instruments which he should provide himself with are,—

A Graduated Straight Ruler or Scale for drawing straight lines and measuring lengths.

A Pair of Compasses with a Pencil for describing circles.

A Pair of Dividers for transferring distances or lengths.

A Protractor for measuring angles.

A Set Square for drawing parallels and perpendiculars.

Straight lines may be drawn, measured and divided with the help of a scale and dividers; circles described with compasses; parallels and perpendiculars drawn with set squares (see Notes to Problems 5 and 6); and angles measured with protractors.

It may be observed here that a right angle or rather the arc of the circle on which it stands, is supposed to be divided into 90 equal parts, and each of these is called a *degree*; so that a right angle is represented by 90°, half a right angle by 45°, and a third part of a right angle by 30°.

In working out numerical examples in Geometry, the student should verify his results by measurement.

5. It should be borne in mind that the points, lines, angles, and figures, referred to in any proposition in Books I, II and III, are supposed to lie in one plane.

I. INTERSECTING STRAIGHT LINES.

THEOREM 1.

If at a point in a straight line, two other straight lines on opposite sides of it meet and are in the same straight line, the two adjacent angles which they on either side of them make with the first mentioned straight line, are together equal to two right angles.

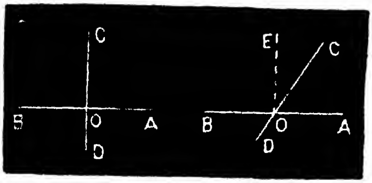


FIG. 1 FIG. 2.

Let the st. lines OA, OB on opposite sides of the st. line CD meet at the pt. O and be in the same st. line;

then $\angle$s AOC and COB together = 2 rt. $\angle$s.

If $\angle$AOC = $\angle$COB as in Figure 1,

then each of them is a rt. $\angle$ (Definition 10),

and $\angle$AOC + $\angle$COB = 2 rt. $\angle$s.

If $\angle$AOC and COB be not equal, as in Fig. 2,

suppose OE $\perp$ AB.

Then $\angle$AOC + $\angle$COB = $\angle$AOC + $\angle$COE + $\angle$EOB,

and $\angle$AOE + $\angle$EOB = $\angle$AOC + $\angle$COE + $\angle$EOB;

$\therefore$ $\angle$AOC + $\angle$COB = $\angle$AOE + $\angle$EOB (Axiom 1),

= 2 rt. $\angle$s (Def. 10).

Similarly it may be shown that

$\angle$AOD + $\angle$DOB = 2 rt. $\angle$s.

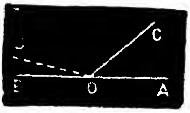
COROLLARY 1. From this it is manifest that the four angles which two intersecting straight lines make, are together equal to four right angles.

COR. 2. When several straight lines meet at a point, the consecutive angles they make, taken all together, are equal to four right angles.

NOTE. Each of the two angles AOC and COB is called the *supplement* of the other, and the two angles are said to be *supplementary* to each other.

THEOREM 2.

If at a point in a straight line two other straight lines on opposite sides of it meet and make the adjacent angles on the same side of them together equal to two right angles, these two straight lines are in the same straight line.



Let the st. lines OA, OB meet the st. line OC at O on opposite sides of it, and make $\angle$s AOC and COB together equal to 2 rt. $\angle$s;
then OA and OB are in the same st. line.

For, if not, let AO produced lie as CD.

Then $\angle$AOC + $\angle$COD = 2 rt. $\angle$s. (Theor. 1).

But $\angle$AOC + $\angle$COB = 2 rt. $\angle$s (by hypothesis):

$\therefore$ $\angle$AOC + $\angle$COD = $\angle$AOC + $\angle$COB (Axiom 1).

and $\therefore$ taking $\angle$AOC from both these equals,

$\angle$COD = $\angle$COB. (Axiom 3)

the less equal to the greater, which is absurd.

Therefore OD must coincide with OB,
or OA and OB must be in the same st. line.

NOTE 1. This proposition is the *converse* of the preceding; and the method of demonstration used is called the *indirect* method, which consists in shewing that every possible assumption other than that of the truth of the proposition to be proved, leads to an absurd result.

NOTE 2. Any *two* points may be joined by a straight line; but any *three* points are not *necessarily* in a straight line.

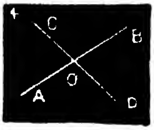
The three pts. A, O, and B are in a st. line only when they are so situated that any st. line OC being drawn through the intermediate pt., the $\angle$s AOC and COB together = 2 rt. $\angle$s.

When three or more points are in the same straight line, they are said to be **collinear**.

Any two straight lines in a plane which are not parallel, meet in a point; but any three or more straight lines which are not parallel may not meet in the same point. When they do, they are said to be **concurrent**.

THEOREM 3.

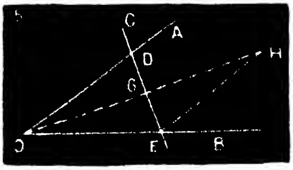
If two straight lines cut one another, the opposite angles are equal.



Let the st. lines AOB and COE cut one another in O;
then $\angle$AOC $=$ $\angle$BOD, and $\angle$AOD $=$ $\angle$BOC.
For $\angle$AOC $+$ $\angle$COB $=$ 2 rt. $\angle$s,
and $\angle$BOD $+$ $\angle$COB $=$ 2 rt. $\angle$s (Theor. 1);
$\therefore$ $\angle$AOC $+$ $\angle$COB $=$ $\angle$BOD $+$ $\angle$COB;
and taking the $\angle$COB from these equals,
$\angle$AOC $=$ $\angle$BOD.
Similarly $\angle$AOD $=$ $\angle$BOC.

THEOREM 4.

If a straight line falls on two intersecting straight lines, it makes the alternate angles unequal, the angle on that side of it on which the two intersecting lines meet being less than the angle on the other side.



Let the st. line CE fall on the intersecting st. lines OA, OB;
then $\angle$ODE $<$ $\angle$DEB, and $\angle$OED $<$ $\angle$EDA.
Suppose DE bisected in G, and OG produced to H so that GH $=$ OG. Join EH.
Place $\triangle$ EGH (reversed) on $\triangle$ DGO so that the pt. G of the one $\triangle$ may be on the pt. G of the other, and the side GE on the side GD:

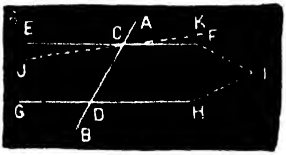
then the pt. E shall fall on the pt D, $\because$ GE = GD.
And GE falling on GD, GH shall fall on GO,
$\because$ $\angle$EGH = $\angle$DGO (Theor. 3),
and the pt. H shall be on the pt O, $\because$ GH = GO.
And $\because$ pts. E and H fall on pts. D and O
$\therefore$ EH shall fall on DO. (Axiom 10).
Thus $\triangle$EGH coincides with $\triangle$DGO, and $\angle$GEH with $\angle$GDO,
and $\therefore$ $\angle$GDO = $\angle$GEH.
But $\angle$GEH < $\angle$DEB,
$\therefore$ $\angle$GDO, that is, $\angle$ODE < $\angle$DEB.
Similarly $\angle$OED < $\angle$EDA.

II. Parallel Straight Lines.

THEOREM 5.

I. *If a straight line falling on two other straight lines, makes the alternate angles equal, these two straight lines are parallel.*

II. *Conversely, if a straight line falls on two parallel straight lines, it makes the alternate angles equal.*



I. Let the st. line AB fall on the st. lines EF, GH
so that $\angle ECD = \angle CDH$;
then EF $\parallel$ GH.
For if not, let EF and GH meet in I.
Then $\angle CDH < \angle ECD$ (Theor. 4)
which is impossible,
$\because \angle CDH = \angle ECD$ by hypothesis.
Hence EF, GH cannot meet in the direction of I.

Similarly it may be shown that they cannot meet in the opposite direction.

They are therefore parallel.

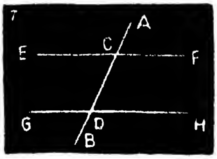
II. Let the st. lines EF, GH be parallel;
then $\angle ECD = \angle CDH$.
For if not, one of them, $\angle ECD >$ the other, $\angle CDH$.
Suppose $\angle JCD = \angle CDH$ (Postulate 8).
Then st. line JCK $\parallel$ GH by the theorem just proved.
And $\because$ EF $\parallel$ GH,
$\therefore$ both ECF and JCK $\parallel$ GH,
which is impossible (Axiom 12).

Hence $\angle$s ECD and CDH are not unequal, that is, they are equal.

THEOREM 6.

I. *If a straight line falls on two parallel straight lines, it makes the exterior angle equal to the interior opposite angle on the same side of it, and the two interior angles on the same side of it together equal to two right angles.*

II. *Conversely, if a straight line falling on two other straight lines makes the exterior angle equal to the interior opposite angle on the same side of it, or the two interior angles on the same side of it together equal to two right angles, the two straight lines are parallel.*



I. Let the st. lines EF, GH be parallel,
and let the st. line AB fall on them;
then $\angle BDH = \angle BCF$,
and $\angle BCF + \angle ADH = 2$ rt. $\angle$s.
For, $\because EF \parallel GH$,
$\therefore \angle BCF = \angle ADG$ (Theor. 5)
$= \angle BDH$ (Theor. 3).
Again, $\because \angle BCF = \angle BDH$,
$\therefore \angle BCF + \angle ADH = \angle BDH + \angle ADH$
$= 2$ rt. $\angle$s. (Theor. 1).

II. Let $\angle BDH = \angle BCF$
or let $\angle BCF + \angle ADH = 2$ rt. $\angle$s;
then $EF \parallel GH$
For, $\because \angle BCF = \angle BDH = \angle GDC$ (Theor. 3),
$\therefore EF \parallel GH$ (Theor. 5).
Again $\because \angle BCF + \angle ADH = 2$rt. $\angle$s $= \angle ADG + \angle ADH$ (Theor. 1),
$\therefore$ taking $\angle ADH$ from both,
$\angle BCF = \angle ADG$,
and $\therefore EF \parallel GH$. (Theor. 5).

NOTE. When a straight line falls on two others, if the latter are parallel, then

(1) the alternate angles are equal,

(2) the exterior angle is equal to the interior opposite angle, and

(3) the two interior angles on the same side of the cutting line are supplementary.

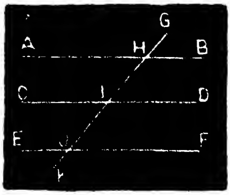
And conversely, if any one of these conditions is satisfied, then the lines are parallel.

The first case is proved independently, and the other two are deduced from it.

It will be seen that there are two pairs of exterior angles and two pairs of interior angles, and the two angles of each pair are supplementary. It will also be observed that the four interior angles taken in pairs alternately, constitute two pairs of alternate angles.

THEOREM 7.

If two straight lines are parallel to the same straight line, they are parallel to one another.



Let the st. lines AB, CD be both parallel to EF;

then AB ‖ CD.

For let any st. line GHIJK fall on the three st. lines.

Then ∵ AB ‖ EF,

∴ ∠AHK = ∠GJF (Theor. 5).

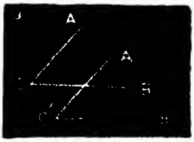
Again ∵ CD ‖ EF

∴ ∠GID = ∠GJF (Theor. 6)

Hence ∠AHK = ∠GID (Axiom 1).

and ∴ AB ‖ CD. (Theor. 5).

Cor. If two intersecting straight lines are respectively parallel to two others, the two pairs of lines contain equal angles.



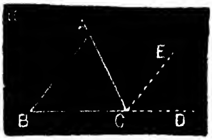
This is clear from the Figure, where

$$\angle C = \angle \text{ between BC and C'A'}$$
$$= \angle C' \text{ (Theor. 6).}$$

III. Relations among the Angles and Sides of a Triangle.

THEOREM 8.

If three straight lines intersect so as to form a triangle, the three interior angles are together equal to two right angles.



Let the three st. lines AB, BC, CA, form the $\triangle ABC$;
then $\angle CAB + \angle ABC + \angle BCA = 2$ rt. $\angle$s.

Produce BC to D and suppose CE $\parallel$ AB.

Then $\because$ CE $\parallel$ AB,

$\therefore \angle CAB = \angle ACE$ (Theor. 5),

and $\angle ABC = \angle ECD$ (Theor. 6);

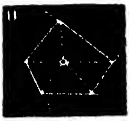
$$\therefore \angle CAB + \angle ABC + \angle BCA = \angle ACE + \angle ECD + \angle BCA$$
$$= \angle DCA + \angle BCA$$
$$= 2 \text{ rt. } \angle\text{s. (Theor. 1).}$$

Cor. 1. Any two angles of a triangle are together less than two right angles.

Note. I. Hence if one angle of a triangle be obtuse or right, the other two angles must both be acute.

Cor. 2. If one side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles, and is greater than either of them.

Cor. 3. All the interior angles of any rectilineal figure together with four right angles, are equal to twice as many right-angles as the figure has sides.



Let us take any rectilineal figure of *n* sides;

then all the interior $\angle$s + 4 rt. $\angle$s = $2n$ rt. $\angle$s.

Take any pt. O within the figure, and join it with all the angular points of the figure. Then the figure is divided into $n\,\triangle$s.

Now the interior $\angle$s of the $n\,\triangle$s = $2n$ rt. $\angle$s.

But interior $\angle$s of the $n\,\triangle$s = interior $\angle$s of the fig. + $\angle$s at O;

and the $\angle$s at O = 4 rt. $\angle$s (Theor. 1, Cor. 2);

$\therefore$ the interior $\angle$s of the fig. + 4 rt. $\angle$s = $2n$ rt. $\angle$s.

Cor. 4. If the sides of any rectilineal figure* which has no re-entrant angle are produced in order, the sum of all the exterior angles is equal to four right angles.

Suppose the figure to have n sides.

Then all the interior $\angle$s + all the exterior $\angle$s = $2n$ rt. $\angle$s.

and all the interior $\angle$s + 4 rt. $\angle$s = $2n$ rt. $\angle$s;

$\therefore$ all the exterior $\angle$s = 4 rt. $\angle$s.

NOTE 2. Theorems 8 and 6 show that when a straight line falls on two other straight lines, the sum of the two interior angles on either side of it is equal to, or less than, or greater than, two right angles, according as these two straight lines are parallel, or are convergent or divergent on that side, the defect or excess being equal to the angle between the two straight lines. And if we regard the angle between two parallel straight lines as zero, the same truth may be shortly stated thus:—When a straight line falls on two other straight lines, the difference between the sum of the two interior angles on either side of it and two right angles is equal to the angle between those two straight lines.

Cor. 5. Corollary 3 enables us to determine the magnitude of an interior angle of any *regular* (that is equilateral and equiangular) rectilineal figure thus:—

Let the figure have n sides.

Then an interior $\angle = \frac{1}{n} \times (2n-4)$ rt. $\angle$s

$= (2 - \frac{4}{n})$ rt. $\angle$s

$= \frac{2}{3}$ of a rt. $\angle$	if $n = 3$.
or $= 1$ rt $\angle$	if $n = 4$,
or $= \frac{6}{5}$ of a rt. $\angle$	if $n = 5$,
or $= \frac{4}{3}$ of a rt. $\angle$	if $n = 6$,
or $= \frac{10}{7}$ of a rt. $\angle$	if $n = 7$,
or $= \frac{3}{2}$ of a rt. $\angle$	if $n = 8$,
&c.	&c.

And hence, $\because$ the angular space round a pt. = 4 rt. $\angle$s, equiangular triangles (6 in number), squares (4 in number), and

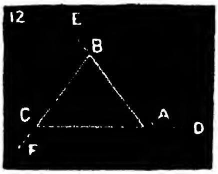
regular hexagons (3 in number), are the only regular figures that can exactly fill such space, 3 regular figures of 7 or more sides filling up more than 4 rt. $\angle$s ($\because 3 \times \frac{10}{7} = 4\frac{2}{7}$ &c, &c.) and 2 such figures being insufficient ($\because$ they fill up angular space that is less than 4 rt. $\angle$s).

NOTE 3. It may be observed that the bee makes the cells of her hive in the shape of regular hexagonal prisms, there being thus no loss of space about any point of junction; and economy of space is further secured by reason of the hexagon approaching more nearly the roundness of the cylindrical larvæ for the abode of which the cells are intended, than the other two regular figures mentioned above. We may therefore truly say of the bee,

"How skillfully she builds her cell!"

NOTE 4. It should be borne in mind that the rectilineal figures referred to above are *plane* figures.

ANOTHER PROOF OF THEOREM 8. The following demonstration of this important theorem, first given by Playfair, deserves the student's careful consideration.

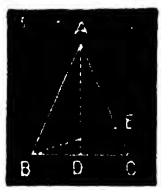


Let ABC be a △.
Produce CA, AB, and BC to D, E, and F respectively.
Let AD rotate about A through ∠DAB,
that is, till it coincides with AB;
and then translate AD along AB till A comes to B.
Now let AD in this altered position rotate about B through ∠EBC,
that is, till it coincides with BC;
and then translate AD along BC till A comes to C.
Next let AD in this altered position rotate about C through ∠ FCA,
that is. till it coincides with CA;
and lastly, translate AD along CA till A and AD
coincide with their initial positions.
Thus by rotation through ∠DAB + ∠EBC + ∠FCA,
and by translation,
AD comes back to its initial position.
And as rotation is independent of translation,
and the total rotation necessary to bring a st. line back
to its initial position, is rotation through 4rt. ∠s,
the total rotation of AD or ∠DAB + ∠EBC + ∠FCA,
that is, the sum of the exterior ∠s of the △ABC = 4rt. ∠s.
But the sum of the exterior ∠s + sum of the interior ∠s = 6rt. ∠s;
∴ sum of the interior ∠s = 2rt. ∠s.

THEOREM 9.

I. *If two sides of a triangle are equal, the angles opposite to them are also equal.*

II. *Conversely, if two angles of a triangle are equal, the sides opposite to them are also equal.*



I. Let the sides AB, AC of $\triangle ABC$ be equal;
then $\angle ACB = \angle ABC$.
For suppose $\angle BAC$ bisected by AD,
and suppose $\triangle ABC$ folded about AD.
Then $\because$ $\angle CAD = \angle BAD$, $\therefore$ AC shall fall on AB;
and $\because$ AC = AB, $\therefore$ pt. C shall fall on pt. B,
and side DC on side DB (Axiom 10).
Thus $\triangle$ ADC coincides with $\triangle$ ADB and $\angle ACB$ with $\angle ABC$
and $\therefore$ $\angle ACB = \angle ABC$, (Axiom 9).

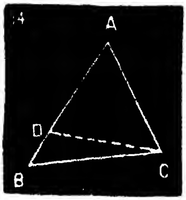
II. Next let $\angle ACB = \angle ABC$;
then AB = AC.
For if not, one of them > the other.
Suppose AC > AB, and AE = AB.
Then $\angle AEB = \angle ABE$ (from what is just proved).
But $\angle AEB > \angle ACB$ (Theor. 8, Cor. 2),
$\therefore$ $\angle ABE > \angle ACB$;
and $\because$ $\angle ABC > \angle ABE$,
$\therefore$ $\angle ABC > \angle ACB$, which is absurd,
being contrary to the hypothesis.
Hence AB and AC are not unequal, that is AB = AC.

Cor. Hence every equilateral triangle is also equiangular, and conversely, every equiangular triangle is also equilateral.

THEOREM 10.

I. *If one side of a triangle is greater than another, the angle opposite to the former is greater than the angle opposite to the latter.*

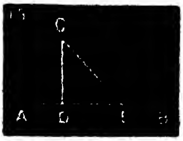
II. *Conversely, if one angle of a triangle is greater than another, the side opposite to the former is greater than the side opposite to the latter.*



I. Let the side AB of the $\triangle$ABC be greater than AC;
then $\angle$ACB $>$ $\angle$ABC.
Suppose AD = AC. Join CD.
Then $\angle$ACD = $\angle$ADC (Theor. 9).
But $\angle$ACB $>$ $\angle$ACD and $\therefore >$ $\angle$ADC,
and $\angle$ADC $>$ $\angle$ABC (Theor. 8, Cor. 2);
$\therefore$ $\angle$ACB $>$ $\angle$ABC.

II. Let $\angle$ACB be greater than $\angle$ABC;
then AB $>$ AC.
For if not, AB either = or $<$ AC.
But AB cannot be equal to AC,
for then $\angle$ACB would be equal to $\angle$ABC which is not the case;
nor can AB be less than AC,
for then $\angle$ACB would be less than $\angle$ABC which is not the case.
Hence AB $>$ AC.

COR. Of all straight lines that can be drawn to a given straight line from a given point without it, the perpendicular is the shortest.

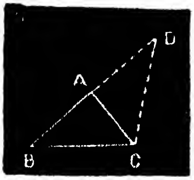


For if CD $\perp$ AB and CE any other st. line,
$\because$ $\angle$CDE is a rt. $\angle$ and $\therefore >$ $\angle$CED
(Theor. 8, COR. 1),
$\therefore$ CE $>$ CD.

NOTE. From Theorems 9 and 10 it follows that if one side of a triangle is greater than, equal to, or less than another, the angle opposite to the former, is greater than, equal to, or less than the angle opposite to the latter.

THEOREM 11.

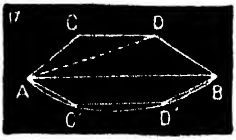
Any two sides of a triangle are together greater than the third side.



Let ABC be a $\triangle$ of which AB, AC are any two sides;
then $AB+AC>BC$.

Produce BA to D and suppose $AD=AC$. Join CD.
Then $\because AD=AC, \therefore \angle ACD=\angle ADC$ (Theor. 9).
But $\angle BCD>\angle ACD, \therefore \angle BCD>\angle ADC$,
and $\therefore$ BD or $BA+AD>BC$ (Theor. 10).
But $AD=AC$;
$\therefore BA+AC>BC$.

Cor. A straight line is the shortest distance between any two points. This is really evident. But if necessary it may be proved thus:—



Let A and B be any two pts.,
joined by the st. line AB and the crooked line ACDB or curve AC′D′B.

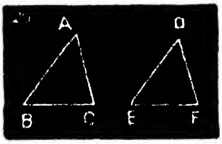
Then $AC+CD>AD$, and $AD+DB>AB$;
$\therefore AC+CD+DB>AB$.
Similarly $AC'+C'D'+D'B>AB$.

And by taking on the curved line a sufficiently large number of points near one another, the length of the curve may be made to differ as little as we please from the crooked line joining those points.

IV. Congruent Triangles.

THEOREM 12.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles contained by those sides equal to each other, then their bases or third sides are equal, the triangles are equal, and their remaining angles are equal, each to each, namely, those to which the equal sides are opposite.



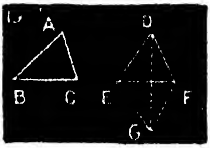
Let ABC and DEF be two $\triangle$s in which
AB = DE, AC = DF, and $\angle$ BAC = $\angle$ EDF;
then BC = EF, $\triangle$ ABC = $\triangle$ DEF,
$\angle$ ABC = $\angle$ DEF, and $\angle$ ACB = $\angle$ DFE.
For, apply the $\triangle$ ABC to the $\triangle$ DEF so that
pt. A falls on pt D, and st line AB on st. line DE;
then B shall fall on E, $\because$ AB = DE.
AC shall fall on DF, $\because$ $\angle$ BAC = $\angle$ EDF,
and C shall fall on F, $\because$ AC = DF.
And $\because$ B and C fall on E and F,
$\therefore$ BC coincides with EF and is equal to it. (Axioms 10 and 9)
Thus the $\triangle$ ABC coincides with the $\triangle$ DEF and is equal to it,
and the $\angle$s ABC, ACB respectively coincide with
the $\angle$s DEF and DFE,
and are equal, each to each. (Axiom 9.)

NOTE 1. Triangles and other figures which can be made to coincide, and are therefore equal in every respect, are said to be **congruent**.

NOTE 2. The student should carefully note the import of the words "each to each, namely, those to which the equal sides are opposite." $\angle$ B to which AC is opposite, is equal to $\angle$ E to which DF, equal to AC, is opposite, and not to $\angle$ F.

THEOREM 13.

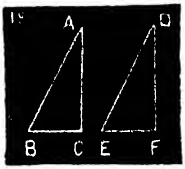
If two triangles have two sides of the one equal to two sides of the other, each to each, and also their bases equal, the angle contained by the two sides of the one is equal to the angle contained by the two sides of the other; and the triangles are equal in every respect.



Let ABC, DEF be two $\triangle$s in which
AB = DE, AC = DF, and BC = EF;
then $\angle$ BAC = $\angle$ EDF, and the $\triangle$s are congruent.
For apply the $\triangle$ ABC to the $\triangle$ DEF so that
B falls on E, BC on EF,
and $\triangle$ ABC on the side of EF opposite to D;
then C falls on F, $\because$ BC = EF.
Let AB, AC have the positions GE, GF. Join DG.
Then $\because$ DE = AB = GE,
$\therefore$ $\angle$ EGD = $\angle$ EDG (Theor. 9);
and $\because$ DF = AC = GF,
$\therefore$ $\angle$ FGD = $\angle$ FDG (Theor. 9);
$\therefore$ adding these equals,
$\angle$ EDF = $\angle$ EGF = $\angle$ BAC,
and $\therefore$ the two $\triangle$s ABC and DEF are congruent (Theor. 12).

THEOREM 14.

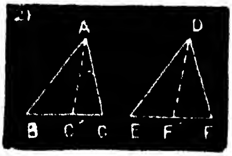
If two triangles have two angles of the one equal to two angles of the other, each to each, and have a side of the one, whether adjacent to the equal angles or opposite to one of them, equal to a corresponding side of the other, the two triangles are equal in every respect.



Let ABC and DEF be two $\triangle$s in which
$\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$,
and either $BC = EF$, or $BA = ED$;
then the $\triangle$s ABC and DEF are equal in every respect.
First suppose $BC = EF$.
Apply the $\triangle$ ABC to the $\triangle$ DEF so that
B may fall on E and BC on EF;
then C shall fall on F, $\because$ $BC = EF$,
BA shall fall on ED, $\because$ $\angle B = \angle E$,
and CA shall fall on FD, $\because$ $\angle C = \angle F$;
and A shall fall on D, $\because$ BA and CA fall on ED and FD,
and any other position of A would make BA and ED or CA and FD, or both pairs of lines, have a common segment, or common segments, which is impossible (Axiom 10).
Thus the two $\triangle$s ABC and DEF coincide,
and are equal in every respect (Axiom 9).
Next suppose $BA = ED$.
Then $\because$ $\angle B + \angle C + \angle A = 2$rt. $\angle$s $= \angle E + \angle F + \angle D$ (Theor. 8),
and $\angle B + \angle C = \angle E + \angle F$, (Hyp.),
$\therefore$ $\angle A = \angle D$,
so that this case becomes similar to the first case,
and $\therefore$ the two $\triangle$s are congruent.

THEOREM 15.

If two triangles have two sides of the one equal to two sides of the other, each to each, and their angles opposite to one pair of equal sides equal, their angles opposite to the other pair of equal sides are either equal or supplementary.



Let ABC, (or ABC′) and DEF be two Δs in which
AB = DE, AC (or AC′) = DF, and $\angle$ ABC = $\angle$ DEF;
then $\angle$ ACB (or $\angle$ AC′B) = $\angle$ DFE (or is supplementary to it).
Apply ΔABC to ΔDEF so that B falls on E and BC on EF,
then BA shall fall on ED, $\because$ $\angle$B = $\angle$E,
and A shall fall on D, $\because$ BA = ED;
and the side AC will either fall on DF,
or, if it is the dotted line AC′, it will have the position DF′.
In the former case $\angle$ACB = $\angle$DFE;
and in the latter, $\angle$AC′B = $\angle$DF′E,
that is, $\angle$AC′B is supplementary to $\angle$DF′F;
but $\angle$DF′F = $\angle$DFE, $\because$ DF = AC′ = DF′;
$\therefore$ $\angle$AC′B is supplementary to $\angle$ DFE.

NOTE. Theorems 12, 13, 14, and 15 relate to the equality of two triangles in every respect, which follows, if, subject to the exceptions presently to be noticed, three out of the six parts, namely, the three sides and the three angles, of one triangle are respectively equal to the corresponding parts of another. The following are the different cases that arise:—

I (*a*). The equal parts being two sides and the angle adjacent to them, the triangles are congruent, as shown in Theor. 12.

(*b*) The equal parts being two sides and an angle opposite to one of them, the triangles are either congruent or the angles opposite to the other pair of equal sides are supplementary, as shown in Theor. 15.

II. The equal parts being two angles and a corresponding side, the triangles are congruent, as shewn in Theor. 14.

III. The equal parts being the three sides, the triangles are congruent, as shewn in Theor. 13.

IV. The equal parts being the three angles, the triangles may not be congruent, as may be seen from the annexed Figure, where BC, B_1C_1, B_2C_2 are parallel, and the Δs ABC, AB_1C_1, AB_2C_2 are equiangular (Theor. 6).

V. A CASE OF NON-CONGRUENT TRIANGLES.

THEOREM 16.

I. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the angles included by those sides unequal, the base of the triangle having the greater included angle is greater than the base of the other.*

II. *Conversely, if two triangles have two sides of the one equal to two sides of the other, each to each, but their bases unequal, the angle included by the two sides of the triangle having the greater base is greater than the corresponding angle of the other.*

I. Let ABC, DEF be two $\triangle$s in which

$AB=DE$, $AC=DF$, but $\angle BAC > \angle EDF$;

then $BC>EF$.

Let DE be not greater than DF.

Suppose $\angle EDG=\angle BAC$, and $DG=DF$ or AC.

Join EG and FG and let EG cut DF in H.

Then $\because$ DE is not greater than DF or DG.

$\therefore \angle DGE$ is not greater than $\angle DEG$ (Theor. 10).

But $\angle DHG > \angle DEG$ (Theor. 8, Cor. 2).

$\therefore \angle DHG > \angle DGE$, and $\therefore$ DG or $DF > DH$,

that is, H falls above F.

Now $\because DG=DF$, $\therefore \angle DFG=\angle DGF$.

And $\angle EFG > \angle DFG$ or $\angle DGF$ and $\therefore > \angle EGF$,

$\therefore EG>EF$ (Theor. 10).

But $\because$ in the $\triangle$s ABC and DEG, $AB=DE$, $AC=DG$

and $\angle A=\angle EDG$,

$\therefore BC=EG$ (Theor. 12). Hence $BC> EF$.

II. If in the $\triangle$s ABC, DEF, $AB=DE$, $AC=DF$ but $BC>EF$,

then $\angle BAC> \angle EDF$.

For, if not, $\angle BAC$ either $=$ or $<\angle EDF$.

But $\angle BAC$ is not equal to $\angle EDF$.

$\because$ in that case $BC=EF$ which is contrary to the hypothesis.

Nor is $\angle BAC < \angle EDF$,

$\because$ in that case $BC < EF$ which is contrary to the hypothesis.

$\therefore \angle BAC > \angle EDF$.

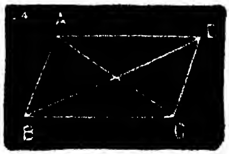
NOTE. Theorems 12 and 16 may be taken together and enunciated thus :—

If two triangles have two sides of the one equal to two sides of the other, each to each, the third side of the one will be greater than, equal to, or less than the third side of the other, according as the angle contained by the other two sides of the former is greater than, equal to, or less than the angle contained by the other two sides of the latter.

VI. PARALLELOGRAMS.

THEOREM 17.

The opposite sides and angles of a parallelogram are equal, and each diagonal bisects it.



Let ABCD be a parm., and AC, BD its diagonals;
then AB = CD, AD = BC, $\angle$ BAD = $\angle$ BCD, $\angle$ ABC = $\angle$ ADC,
$\triangle$ ABD = $\triangle$ CDB, and $\triangle$ ABC = $\triangle$ CDA.

For $\because$ AB $\parallel$ CD, $\therefore$ $\angle$ ABD = $\angle$ CDB (Theor. 5);
and $\because$ AD $\parallel$ BC, $\therefore$ $\angle$ ADB = $\angle$ CBD (Theor. 5).
Thus, in the two $\triangle$s ABD, CDB,
$\angle$ ABD = $\angle$ CDB, $\angle$ ADB = $\angle$ CBD, and BD is common to both,
$\therefore$ AB = CD, AD = CB, $\angle$ BAD = $\angle$ BCD,
$\triangle$ ABD = $\triangle$ CDB, (Theor. 14).
Again $\because$ $\angle$ ABD = $\angle$ CDB, and $\angle$ CBD = $\angle$ ADB,
$\therefore$ the whole $\angle$ ABC = the whole $\angle$ ADC.
Similarly it may be shewn that $\triangle$ ABC = $\triangle$ CDA.

COR. 1. The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are equal and parallel.

Let AB and CD (in the above fig.) be equal and parallel;
then AD and BC are equal and parallel.
Join AC. Then in the two $\triangle$s BAC, DCA,
AB = CD, AC is common and $\angle$ BAC = $\angle$ DCA;
$\therefore$ AD = BC, and $\angle$ ACB = $\angle$ CAD (Theor. 12),
and $\therefore$ AD $\parallel$ BC (Theor. 5).

COR. 2. If one angle of a parallelogram is a right angle, all its angles are right angles.

For $\angle$ BAD + $\angle$ ABC = 2 rt. $\angle$s (Theor. 6);
$\therefore$ if $\angle$ BAD = a rt. $\angle$, then $\angle$ ABC also = a rt. $\angle$.
And the other two $\angle$s which are opposite and therefore equal to these are also rt. $\angle$s.

NOTE. The rectangle contained by AB and BC, is shortly called the rectangle AB, BC, and is named by two letters placed at its opposite angles.

Cor. 3. If three or more parallel straight lines make equal intercepts on any straight line cutting them, then their intercepts on any other straight line that cuts them are also equal.

Let AB, CD, EF be three parallel st. lines,
and let their intercepts IJ, JK on GH be equal;
then their intercepts NO, OP on LM are also equal.
Suppose BOQ ‖ GH. Then IJOB and JKQO are parms.,
and ∴ OB = IJ = JK (by hypothesis) = OQ;
and ∠ OBN = ∠ OQP, and ∠ ONB = ∠ OPQ (Theor. 5);
∴ from △s ONB and OPQ, ON = OP (Theor. 14).

Cor. 4. Parallel straight lines are everywhere equidistant. For if perpendiculars are drawn upon one of them from any two points in the other, a parallelogram will be formed of which the two perpendiculars will be opposite sides, and so they will be equal.

VII. AREAS OF PARALLELOGRAMS AND TRIANGLES.

THEOREM 18.

Parallelograms upon the same base and between the same parallels are equal in area.

Let parms. ABCD and EBCF be upon the same base BC, and between the same parallels BC, AF;
then parm. ABCD = parm. EBCF.
For ∵ ABCD and EBCF are parms.,
∴ AB = DC, and BE = CF (Theor. 17);
and ∵ AB ∥ DC and BE ∥ CF,
∴ ∠ ABE = ∠ DCF (Theor. 7, Cor.);
∴ △ABE = △DCF (Theor. 12).
Now taking these equal △s successively from the figure ABCF, the two remainders are parm. ABCD and parm. EBCF;
∴ parm. ABCD = parm. EBCF (Axiom 3).

NOTE 1. The two parallelograms ABCD and EBCF are equal, not in every respect, but only in area. This proposition is the first instance of the equality of two figures in area only, as distinguished from equality in every respect.

From the above proof it will be seen that either of the two parallelograms can be cut into parts which taken together will exactly fill up the space occupied by the other.

NOTE 2. If parallelograms on the same base are of the same altitude, they are equal. For they may be placed on the same side of the base, and then they will be between the same parallels, because if the *altitudes*, that is, perpendiculars to the base from points in the opposite side, are drawn on the same side of the base, they being equal and parallel, the straight line joining their further extremities will be parallel to the base (Theor. 17, Cor. 1).

THEOREM 19.

Parallelograms upon equal bases and between the same parallels are equal in area.

Let parms. ABCD and EFGH be upon equal bases BC and FG, and between the same parallels AH and BG;
then parm. ABCD = parm. EFGH.
Join EB, CH.

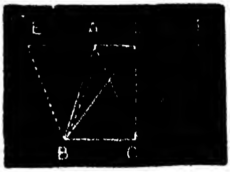
Then ∵ BC = FG = EH (Theor. 17) and BC ∥ EH,
∴ BE ∥ CH. (Theor. 17, Cor. 1) and EBCH is a parm.
And parm. ABCD = parm EBCH (Theor. 18)
= parm. EFGH (Theor. 18).

NOTE. If parallelograms on equal bases are of the same altitude, they are equal. For, as pointed out in Note 2 to the preceding theorem, they may be placed between the same parallels.

THEOREM 20.

I. *Triangles upon the same base and between the same parallels are equal in area.*

II. *Conversely, equal triangles upon the same base are between the same parallels.*



I. Let $\triangle$s ABC, DBC be upon the same base BC, and between the same parallels AD and BC;
then $\triangle$ ABC $=$ $\triangle$ DBC.

Suppose BE $\parallel$ AC, and CF $\parallel$ BD, and complete the parms. EBCA, FCBD.

Then $\because$ parm. EBCA $=$ parm. FCBD (Theor. 18),
and $\triangle$ ABC $= \frac{1}{2}$ parm. EBCA,
and $\triangle$ DBC $= \frac{1}{2}$ parm. FCBD (Theor. 17),
$\therefore$ $\triangle$ ABC $=$ $\triangle$ DBC (Axiom 7).

II. Let $\triangle$s ABC and DBC be equal,
then AD $\parallel$ BC.

For if not, suppose DG $\parallel$ BC.

Then $\triangle$ GBC $=$ $\triangle$ DBC $=$ $\triangle$ ABC, which is absurd (Axiom 8);
$\therefore$ AD $\parallel$ BC.

COR. 1. From this and Theorem 17, it is clear that if a parallelogram and a triangle are upon the same base and between the same parallels, the triangle is half of the parallelogram.

COR. 2. From this and Theorem 19, it is clear that triangles upon equal bases and between the same parallels are equal in area.

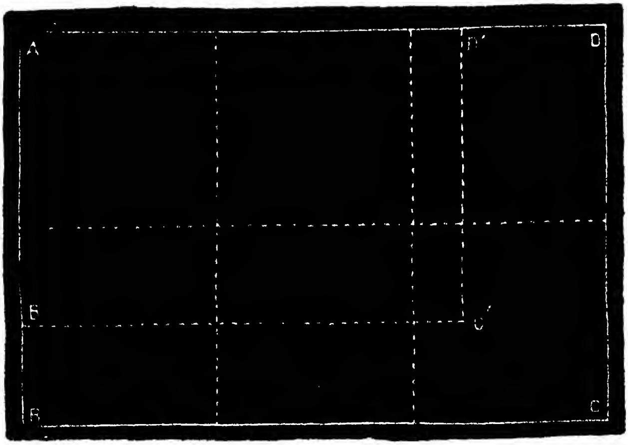
NOTE 1. The above proposition and its converse will be equally true, if for the words "between the same parallels" the words "of the same altitude" are substituted. This will be clear from Note 2 to Theorem 18, it being borne in mind that the altitude of a triangle is the perpendicular on the base from the opposite angle.

NOTE 2. Theorems 18 to 20 help us in expressing areas of parallelograms and triangles numerically.

Any magnitude may be represented numerically by adopting some *definite* magnitude of the *same* kind as the *unit*, and ascertaining what *number* of times it contains the unit. The number thus ascertained will represent the magnitude measured, in this sense, that it will indicate what number of times the magnitude in question contains the unit adopted, so that we may have an idea of the *quantity* of the magnitude, if not of its *form*, by multiplying the unit by the ascertained number.

Thus in measuring lengths we may adopt an inch, or a tenth of an inch, or a foot, as our unit: and then any given length which contains 24 inches, or 240 tenths of an inch, or 2 feet, will be represented by 24 inches, or 240 tenths of an inch, or 2 feet, so that we may have an idea of the *quantity* of the length by multiplying an inch by 24, or a tenth of an inch by 240, or a foot by 2, though these numbers will give us no idea as to the form of the line of which the length is measured, that is, as to whether the line is straight or curved.

In measuring areas, the area of some figure of definite shape and size should be adopted as the unit, and any given area will be represented by the number which indicates how often it contains the unit. For the sake of *simplicity* and *convenience*, the *square* on the *unit of length* adopted is taken as the *unit of area*. That this is a *simple* unit to adopt is clear; that its adoption leads to *convenient* results will be seen presently.



Suppose we have to measure the area of a rectangle ABCD which measures 2 inches and 3 inches along the sides AB and BC respectively, an inch being the linear unit chosen. Dividing AB and BC into 2 and 3

equal parts respectively, and drawing parallels through the points of division as in the figure, we have the rectangle divided into 2 rows of squares, each row containing 3 squares, whereof each is a square on an inch ; so that there are altogether 3×2 square inches in it. The side BC is usually called the *base*, and the side AB, the *altitude* of the rectangle ABCD, so that the area of the rectangle contains *a number* of squares on the linear unit, or *units of area*, equal to the *product* of the number of *linear units* in the base and the number of *linear units* in the altitude. And this is shortly stated by saying that—

The area of a rectangle is equal to the product of the base and the altitude.

As the area of any parallelogram on the same base, and between the same parallels, that is, having the same altitude, is, by Theorem 18, equal to the area of the rectangle, we may also say that,

The area of a parallelogram is equal to the product of its base and altitude.

And as the area of a triangle on the same base, and between the same parallels, that is, of the same altitude, as a parallelogram, is half of that of the parallelogram, we may say that,

The area of a triangle is equal to half the product of its base and its altitude.

The same thing is true if AB and BC involve fractions. Thus if $AB' = 1\frac{1}{2}$ inches, and $B'C' = 2\frac{1}{3}$ inches, the rectangle will contain $2\frac{1}{3} \times 1\frac{1}{2}$ square inches, that is,

2×1 entire squares, in the first horizontal row
$+ 2 \times \frac{1}{2}$ portions of a square in the second ... ...
$+ \frac{1}{3} \times 1$ portion ... ... third vertical row
$+ \frac{1}{3} \times \frac{1}{2}$ ... ... corner.

Hence, generally, if AB and BC contain a and b linear units, the area of the rectangle ABCD contains $a \times b$ superficial units, each being a square on the linear unit ;

or shortly if $AB = a$ and $BC = b$,
rectangle $ABCD = a \times b$,

a very *convenient* formula, which the adoption of the square on the linear unit as the unit of area leads to.

If $a = b$, ABCD is a square $= a^2$

NOTE 3. It will be easily seen from the above figure, that if the three parts into which AD′ is divided are respectively equal to a, b and c, that is, if $AD' = a + b + c$, and $AB' = k$,

$$(a + b + c)k = ak + bk + ck.$$

VIII. Relation between the Square on one Side of a Triangle and those on the other two Sides.

THEOREM 21.

In a right-angled triangle, the square on the side opposite to the right angle is equal to the sum of the squares on the other two sides.

Let ABC be a right-angled $\triangle$ having the rt. $\angle$ BAC;
then sq. on BC$=$sq on AB$+$sq. on AC.

Let BDEC, ABFG, and ACHI be the sqs. on BC, AB, and AC.
Join AD and CF, and suppose AJ $\parallel$ BD or CE.
Then $\because$ $\angle$ BAC is a rt. $\angle$ and so is $\angle$ BAG (Theor. 17, Cor. 2)
$\therefore$ CA and AG are in the same st. line (Theor. 2), and CG $\parallel$ BF.
And $\because$ $\angle$CBD$=$ $\angle$ABF (each being a right $\angle$),
adding $\angle$ABC to both,
$\angle$ABD$=$ $\angle$FBC: and BD$=$BC and BA$=$BF;
$\therefore$ $\triangle$ABD$=$$\triangle$FBC (Theor. 12).
Now rectangle BJ is double of $\triangle$ ABD,
and sq. BG is double of $\triangle$FBC (Theor. 20, Cor. 1);
$\therefore$ sq. BG$=$rect. BJ.
Similarly sq. CI$=$rect. CJ.
And $\therefore$ sq. on AB$+$sq. on AC$=$rect. BJ$+$rect. CJ$=$sq. on BC.

Note 1. This theorem is known as the Theorem of Pythagoras. But it was known to the Hindus in very early times, as we gather from the S'ulva Sutras. See Dr. Thibaut's Paper in the Journal of the Asiatic Society of Bengal, Vol. 44 (1875) p. 227.

The side opposite to the right angle is called the *hypotenuse*.

Note 2. Using the notation in Note 2 to Theorem 20,
$AB^2+AC^2=BC^2$ or if $BC=a$, $CA=b$ and $AB=c$, $a^2=b^2+c^2$.
And if $b=c$, $a^2=2b^2$, or $a=\sqrt{2}\times b$.

Hence the diagonal of a square$=\sqrt{2}\times$ the length of a side.

Now $\sqrt{2}$ is *incommensurable*, that is, it is incapable of being expressed exactly by any number, integral or fractional, though its value may be approximately expressed to any degree of accuracy we please, by carrying on the operation of extracting the square root of 2 to more and more places of decimals.

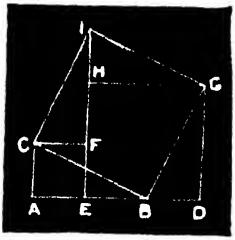
The value of $\sqrt{2}$ as found by calculation is 1·414213............If the side of a square is 1 inch, and we calculate $\sqrt{2}$ to 4 places of decimals, the diagonal will be represented by 1·41 42 inches ; that is, if we regard $\frac{1}{10000}$th of an inch as the unit, the side will be represented by 10000 and the diagonal by 14142, and the difference between this last number and the true length of the diagonal is less then $\frac{1}{10000}$th of an inch. Again, if we calculate $\sqrt{2}$ to 6 places of decimals, that is, regard $\frac{1}{1000000}$th of an inch as the unit, the side and the diagonal will be represented respectively by 1000000 and 1414213, and the difference between this last number and the true length of the diagonal is less than $\frac{1}{1000000}$th of an inch. And in this way, any degree of accuracy can be secured by taking more and more places of decimals, that is, by adopting smaller and smaller units.

Practically then, all magnitudes may be considered as commensurable, necessary accuracy in the case of incommensurable magnitudes being secured by adopting as small a unit as may be required for the purpose.

NOTE 3. Any side of a right-angled triangle may be determine numerically, if the other two sides are given, from the formula.

$$a^2 = b^2 + c^2.$$

ANOTHER PROOF OF THEOREM 21.



Let ABC be a right angled $\triangle$ having the rt. $\angle$A.
Produce AB to D, and suppose BD=AC, and AE=AC.
Then ED=AB.
Let ACFE and EDGH be sqs. on AC and ED.
Then EDGH = sq. on AB.

Produce FH to I, and suppose HI=AC. Join CI, IG, GB.
Then $\triangle$s BDG, IHG, and CFI are congruent with $\triangle$CAB (Theor. 12).

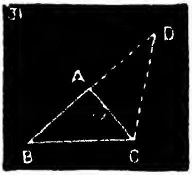
And $\therefore$ CB=BG=IG=CI
Also $\angle$ IGH=$\angle$ BGD, and $\therefore$ $\angle$ IGB=$\angle$ DGH=a rt. $\angle$.
Again $\angle$ CIG=$\angle$ CIF + $\angle$ HIG= $\angle$ CIF + $\angle$ FCI
= a rt. $\angle$ (Theor. 8).
Thus BCIG is a sq. on BC.

And BCIG or sq. or BC=Fig. CBGHF + $\triangle$IHG + $\triangle$CFI
=Fig. CBGHF + $\triangle$BDG + $\triangle$CAB
=sq. EDGH + sq. AEFC
=sq. on AB + sq. on AC.

NOTE. 4. This method of proof shows how the square on the hypotenuse can be cut up into parts (namely, the three parts, figure CBGHF, triangle IHG and triangle CFI) which being re-arranged make up the squares on the other two sides placed in juxtaposition.

THEOREM 22.

If the square on one side of a triangle is equal to the sum of the squares on the other two sides, the angle opposite to the first mentioned side is a right angle.



Let ABC be a $\triangle$ in which sq. on BC = sq. on AB + sq. on AC
then $\angle$ BAC is a rt. $\angle$.

Suppose AD $\perp$ AC and = AB. Join CD.

Then sq. on DC = sq. on AD + sq. on AC (Theor. 21)
= sq. on AB + sq. on AC ($\because$ AD = AB)
= sq. on BC (by hypothesis) ; $\therefore$ DC = BC.

Thus in the two $\triangle$s ABC and ADC,
AB = AD, AC is common, and BC = DC ;
$\therefore$ $\angle$ BAC = $\angle$ DAC (Theor. 13) = a rt. $\angle$.

Note. This proposition is the converse of Theorem 21.

THEOREM 23.

The square on one side of a triangle is equal to, or greater than or less than, the sum of the squares on the other two sides, according as the angle contained by these two sides is a right angle, an obtuse angle or an acute angle, the excess or defect being equal to twice the rectangle contained by either of these two sides and the straight line intercepted between the foot of the perpendicular on it from the opposite angle and the angle contained by these two sides.

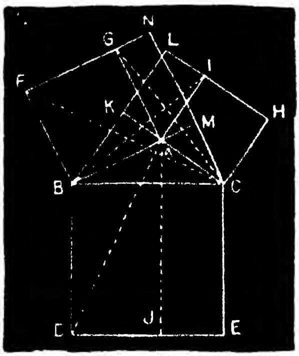


Fig. 1.

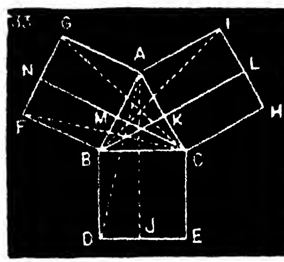


Fig. 2.

Let ABC be a △:

then sq. on BC = or > or < sq. on AB + sq. on AC,

according as ∠ BAC is a rt. ∠ or an obtuse ∠ or an acute ∠;

and in the last two cases, if BK ⊥ CA and CM ⊥ BA;

sq. on BC = sq. on BA + sq. on CA

+ or — twice the rectangle contained by BA, AM, or by CA, AK.

The first case has already been proved in Theorem 21.

In the other two cases, it may be proved as in Theorem 21, that △ ABD = △ FBC;

∴ rectangle BJ = rectangle BN = sq. on BA + or — rectangle MG.

Similarly,

rectangle CJ = rectangle CL = sq. on CA + or — rectangle KI;

∴ adding equals to equals,

rectangle BJ + rectangle CJ or sq. on BC

= sq. on BA + sq. on CA ± rectangle MG ± rectangle KI.

Now $\angle BAG = \angle CAI$ (for each is a rt. $\angle$);

$\therefore$ adding $\angle GAI$ in Fig. 1 or $\angle BAC$ in Fig. 2 to each,

$\angle BAI = \angle GAC$; and $BA = GA$, $AI = AC$:

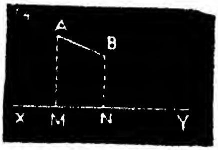
$\therefore \triangle BAI = \triangle GAC$ (Theor. 12);

and $\therefore$ rectangle KI = rectangle MG (being doubles of the equal $\triangle$s):

$\therefore$ sq. on BC = sq. on AB + sq. on AC $\pm$ twice the rectangle MG
or $\pm$ twice the rectangle KI

= sq. on AB + sq. on AC
$\pm$ twice the rectangle contained by BA, AM
or $\pm$ twice the rectangle contained by CA, AK.

NOTE 1. The projection of one straight line on another is that portion of the latter which is intercepted between the feet of the perpendiculars on it from the extremities of the former.



Thus MN in the annexed Figure is the projection of AB on XY.
In the above Figures 1 and 2,
AM is the projection of AG on AB,
and AK is the projection of AB on AC,
for the foot of the perpendicular from A on AB or AC is A.

By using the term *projection* as defined above, we may shorten the enunciation of Theorem 22 thus :—

The square on one side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by these two sides is obtuse, right, or acute, the excess or defect being equal to twice the rectangle contained by either of these sides and the projection on it of the other.

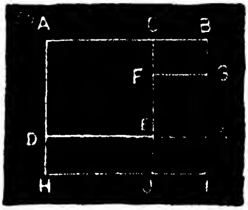
NOTE 2. This Theorem may be deduced from Theorem 21 with the help of Theorem 24 thus :—

$$\begin{aligned} BC^2 &= BM^2 + CM^2 \text{ (Theor. 21)} \\ &= (BA \pm AM)^2 + CM^2 \\ &= BA^2 \pm 2\,BA.\,AM + AM^2 + CM^2 \text{ (Theor. 24, Notes 1 and 2)} \\ &= BA^2 + AC^2 \pm 2BA.\,AM \text{ (Theor. 21).} \end{aligned}$$

IX. Areas of Rectangles and Squares.

THEOREM 24.

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the two parts.



Let the st. line AB be divided into any two parts, AC, CB; then sq. on AB = sq. on AC + sq. on CB + twice rect. AC, CB.

Suppose AHIB, ADEC, CFGB to be the squares on AB, AC, CB.

Produce CE to meet HI in J.

Then ∵ AH = AB, and AD = AC. ∴ DH = CB. And DE = AC.

Hence rect. DJ which is contained by DE, DH = rect. AC, CB.

Again ∵ BI = AB, and BG = BC. ∴ GI = AC; and GF = CB.

Hence rect. GJ which is contained by GI, GF = rect. AC, CB.

Now AHIB = ADEC + CFGB + DJ + GJ;

∴ sq. on AB = sq. on AC + sq. on CB + twice rect. AC, CB.

Cor. 1. If AC = CB, sq. on AB = 4 times sq. on AC.

Cor. 2. If there are two straight lines one of which is divided into any two or more parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

For rect. AH, AC = rect. AD, AC + rect. HD, AC.

Note 1. If AC = a, and BC = b, then AB = $a+b$; and adopting the notation explained in Note 2 to Theorem 20, we have,

$$(a+b)^2=a^2+2ab+b^2,$$

which is the algebraical statement of Theorem 24.

Note 2. If AB = a, and BC = b, then AC = $a-b$; AI = a^2, EI = CG = b^2, AE = $(a-b)^2$; CI = DI = ab;

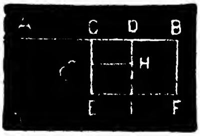
and AE = AI − DI − CK = AI − DI − CI + EI = AI + EI − 2CI,

or $(a-b)^2=a^2-2ab+b^2$.

Note 3. If AC = a, and BC = b, then ∵ AI = AJ + CI and AJ = AE + DJ ∴ $(a+b)^2=a(a+b)+b\,(a+b)$ and $a\,(a+b)=a^2+ab$.

THEOREM 25.

If a straight line is bisected, and also cut unequally internally, the difference between the squares on half the line and on the line between the points of section, is equal to the rectangle contained by the unequal segments.



Let the st. line AB be bisected in C and cut unequally in D; then sq. on CB—sq. on CD=rect. AD, DB.

Suppose CEFB and CGHD to be the squares on CB and CD.

Produce DH to meet EF in I.

Then $\because$ BF=BC=AC,

$\therefore$ rect DF=rect. AC, DB.

Again $\because$ CE=CB, and CG=CD, $\therefore$ GE=DB; and GH=CD;

$\therefore$ rect. GI=rect. CD, DB.

Hence rect. DF+rect. GI=rect. AC, DB+rect. CD, DB;

and $\therefore$ sq. on CB—sq on CD=CF—CH=DF+GI

= rect. AC, DB+rect. CD, DB

=rect. under (AC+CD) and DB

=rect. under AD, DB

COR. Hence, if a straight line is divided into two parts, the rectangle contained by them is the greatest when they are equal.

For rect. AC, CB=CB²=AD. DB+CD²>rect. AD, DB.

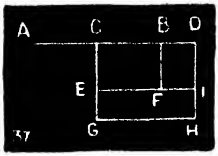
NOTE 1. If AC=CB=a, and CD=b, then AD=$a+b$ and DB=$a-b$, and $a^2-b^2=(a+b)(a-b)$,
which is the algebraical statement of Theorem 25.

NOTE 2. When several magnitudes satisfy certain conditions, if there is one of them which is greater than all the rest, it is said to be a **maximum**; and if there is one of them which is less than all the rest, it is said to be a **minimum**.

Thus the rectangle contained by the two parts of a given straight line is the maximum when the parts are equal. Again of all the straight lines which may be drawn from a given point to a given straight line, the perpendicular is the minimum (Theor. 10, Cor).

THEOREM 26.

If a straight line is bisected, and produced to any point, that is, cut unequally externally, the difference between the squares on half the line and the line between the points of section, is equal to the rectangle contained by the unequal parts.



Let the st. line AB be bisected in C,
and cut unequally externally in, that is produced to, D;
then sq. on CD—sq. on CB=rect. AD, DB.
Suppose CEFB, CGHD to be the squares on CB, CD.
Produce EF to meet DH in I.
Then it may be shown as in Theorem 25, that
rect. DF=rect. AC, DB,
and rect. GI=rect. CD. DB;
so that rect. DF+rect. GI=rect. AC, DB+rect. CD, DB
=rect. under (AC+CD) and DB
=rect. under AD, DB.
And ∴ sq. on CD—sq. on CB=CH—CF=rect. DF+rect. GI
=rect. AD, DB.

COR. The rectangle contained by the sum and difference of any two straight lines is equal to the difference of their squares.

NOTE 1. If AC=CB=a, CD=b, then AD=$a+b$, and DB=$b-a$, and
$$b^2-a^2=(b+a)(b-a)$$
which is the algebraical statement of Theorem 26.

NOTE 2. When a straight line is produced to any point, that point may be regarded as a point of *external* section, the two segments of the line being the distances of its extremities from that point, and one of these segments being thus greater than the whole line.

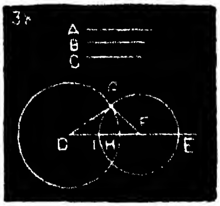
SECTION III. PROBLEMS.

Introductory Remark. The few Problems that follow, are intended to help the student in drawing correct geometrical figures, and to show how with the aid of Postulates 1 to 3, complicated geometrical constructions can be effected.

I. CONSTRUCTION OF TRIANGLES AND ANGLES.

PROBLEM 1.

To construct a triangle having its three sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be three st. lines any two of which are together greater than the third :

it is required to construct a △ having its sides respectively equal to A, B, C.

Take any st. line DE ; make DF equal to A ;
with centre D and radius B describe ⊙ HG ;
and with centre F and radius C describe ⊙IG.
Then the ⊙s must intersect.
For they cannot be wholly without each other,
$\because$ B + C > A or DF ;
nor can either be wholly within the other,
$\because$ A + B > C and A + C > B.
Let the ⊙s intersect in G. Join DG, FG.
Then DFG is the △ required.
For DF = A, DG = B, and FG = C.

NOTE. The condition that any two of the given straight lines together should be greater than the third, must be satisfied in order that a triangle having its sides equal to them may be possible, because any two sides of a triangle are together greater than the third, as is shewn in Theorem 11. And if this condition is not satisfied, the circles in the above figure will not intersect, and the construction of a triangle will not be possible.

PROBLEM 2.

At a given point in a given straight line, to make an angle equal to a given angle.

Let it be required to make at A in st. line AB
an ∠ equal to ∠ CDE.
Take any pt C in DC; with centre D and radius DC
describe ⊙CE cutting DE in E, join CE;
with centre A and radius DC describe ⊙FG cutting AB in F;
with centre F and radius CE describe a ⊙ cutting ⊙ FG in G;
and join AG, GF.

Then ∠ FAG is the ∠ required.
For AF = DC, AG = AF = DC = DE, and FG = CE;
∴ ∠ FAG = ∠CDE (Theor. 13).

Cor. Hence we can construct a triangle with any given parts which determine it.

i. When the given parts are the three sides, the construction is made by Problem 1.

ii. When the given parts are two sides and the contained angle, the construction may be made thus:—

At A in the given side AB make ∠ BAF equal to given ∠ E (Problem 2); make AF equal to given side CD; and join BF.

Then ABF is the △ required.

iii. When the given parts are two angles and a side, the construction may be effected thus :—

Let CAD, EBF be the given ∠s, and GH the given side adjacent to the given angles or opposite to one of them.

First suppose GH to be adjacent to the given ∠s.

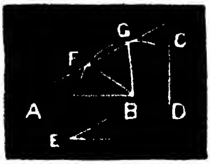
At G and H in GH make ∠s HGK and GHK equal to the given ∠s.

Then GHK is the △ required.

Next suppose the given side to be KG opposite to one of the given ∠s, EBF.

Then the third ∠ to which KG must be adjacent, may be easily found by making at A in CA the ∠ CAI equal to ∠EBF and producing DA to J. For ∠IAJ must be the third ∠, ∵ the three ∠s = 2rt. ∠s. Thus the two ∠s to which KG is adjacent being known, the △ can be made as in the preceding case.

iv. When the given parts are two sides and the angle opposite to one of them, the construction may be made thus :— Let AB, CD be the given sides and ∠E the angle opposite to CD.



At A in BA make ∠BAG equal to ∠E :
with centre B and radius CD describe ⊙FG
cutting AG if possible in F and G ; and join BF, BG.
Then ABF or ABG is the △ required.

There will be two △s, or one only, or none, according as CD > = or < the perpendicular from B on AG, supposing CD < AB, and ∠E is acute.

When ∠E = or > a rt. ∠, CD must be greater than AB, and there will be only one △.

II. Bisection of Angles and Straight Lines.

PROBLEM 3.

To bisect a given angle.

Let BAC be the given ∠ : it is required to bisect it.

With centre A and radius AB (B being any pt. in AB) describe ⊙BC ; with centre B and radius BC describe ⊙CD ; with centre C and radius CB describe ⊙BD ; and join AD, BD and CD. Then AD bisects ∠BAC.

For AB = AC, AD is common, and BD = BC = CD.

in the 2 △s ABD and ACD ; ∴ ∠BAD = ∠CAD (Theor. 13).

Note 1. Hence any angle may be divided into 4, 8, 16, etc. equal parts.

Note 2. If from any pt. D in AD, DE, DF be drawn ⊥ AB, AC, then from the two △s AED, AFD, DE = DF (Theor. 14).

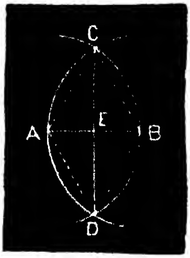
Thus every pt. in AD is equidistant from AB and AC.

*When every point in a given line, straight or curved, satisfies certain given conditions, the line is said to be the **locus** of the point satisfying those conditions.

Cor. **The locus of the point equidistant from two intersecting straight lines is the pair of bisectors of the angles between those lines.**

PROBLEM 4.

To bisect a given straight line.



Let it be required to bisect st. line AB.
With centre A and radius AB describe ⊙ CBD
with centre B and radius BA describe ⊙ CAD;
and join CD cutting AB in E.
Then AB is bisected in E.
For join CA, CB, DA, DB.
Then in $\triangle$s ACD and BCD,
AC = AB = BC, CD is common, and AD = AB = BD,
$\therefore \angle ACD = \angle BCD$ (Theor. 13).
Again in $\triangle$s ACE and BCE,
AC = BC, CE is common, and $\angle ACE = \angle BCE$,
$\therefore$ AE = BE (Theor. 12).

NOTE. Hence a straight line may be divided into 4, 8, 16 etc. equal parts.

III. Drawing of Parallels and Perpendiculars to Straight Lines.

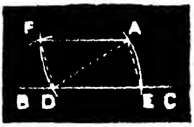
PROBLEM 5.

From a given point to draw a straight line parallel to a given straight line.

Let it be required to draw from A a st. line ∥ BC.
In BC take any pt. D; join AD; and at A in DA
make ∠DAE equal to ∠ADC (Prob. 2).
Then AE ∥ BC.
For, ∵ ∠DAE = ∠ADC, ∴ AE ∥ BC (Theor. 5).

NOTE. In practice, parallels may be conveniently drawn with the help of set squares, as shewn in the figure, where FSD and F′S′A are the two positions of the set square so that ∠F′AD = ∠FDS, and AF′ ∥ BC.

ANOTHER METHOD. In this method, no reference is made to Problem 2.



Let it be required to draw from A a st. line ∥ BC.
In BC take any pt. D, and join AD.
With centre D and radius DA describe a ⊙ cutting BC in E,
with centre A and radius AD describe a ⊙DF; and
with centre D and radius = AE describe a ⊙ cutting ⊙DF in F.
Join AF. Then AF ∥ BC.
For in the △s ADE and DAF,
DE = DA = AF, AD is common, and AE = DF,
∴ ∠ADE = ∠DAF (Theor. 13); and ∴ AF ∥ BC (Theor. 5).

PROBLEM 6.

To draw a straight line perpendicular to a given straight line from a given point in or without it.

I. Let it be required to draw a st. line $\perp$ AB from the pt. C in it.

Make CD equal to CA ;

on AD describe the equilateral $\triangle$ AED (Prob. 1) ; and join EC.

Then EC $\perp$ AB.

For in the two $\triangle$s ACE and DCE,

AC=DC, EC is common, and AE=DE ;

$\therefore$ $\angle$ACE=$\angle$DCE (Theor. 13),

and $\therefore$ CE $\perp$ AB (Def. 10).

II. Let it be required to draw a st. line $\perp$ AB from the pt. C without it.

Take any pt. D on the other side of AB ;

with centre C and radius CD describe the $\odot$ EDF ;

bisect EF in G (Prob. 4) ; and join CG.

Then CG $\perp$ AB.

For in the two $\triangle$s CGE, CGF,

EG=FG, CG is common, and CE=CF ;

$\therefore$ $\angle$CGE=$\angle$CGF (Theor. 13),

and $\therefore$ CG $\perp$ AB.

Cor. Every point in CE (Fig. 1) is equidistant from A and D ; that is, in other words, the locus of the point equidistant from two given points is the perpendicular bisecting the straight line joining them.

Cor. 2. Hence we can describe a square on a given st. line AB thus :—

Draw AC $\perp$ AB and =AB, and draw BD $\parallel$ AC, and CD $\parallel$ AB.

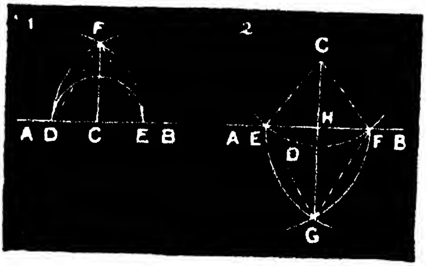
Then evidently ACDB is the square required.

Note. In practice, perpendiculars may be most conveniently drawn with the help of a set square ; and this problem is intended more to show how in theory, without a set square, and with the aid of the ruler and the compasses only, a perpendicular may be drawn, than to furnish a

practical method of drawing a perpendicular. This is the reason why in the foregoing solution, Problems previously solved, namely Problems 1 and 4, are relied upon instead of independent constructions being given.

The same remarks apply to Problem 5.

ANOTHER METHOD. In this method no reference is made to any previous Problem.



I. Let it be required to draw a st. line ⊥ AB from the pt. C in it (Fig. 1.)

In AC take any pt. D,

with centre C and radius CD describe a ⊙ cutting CB in E;

with centre D and radius DE describe the ⊙ EF;

with centre E and radius ED describe the ⊙ DF cutting ⊙ EF in F,

and join CF.

Then CF ⊥ AB.

For in the two △s FCD and FCE, CD = CE,

CF is common, and DF = DE = EF,

∴ ∠FCD = ∠FCE (Theor. 13), and ∴ each is a rt. ∠. (Df. 10).

II. Let it be required to draw a st. line ⊥ AB from the pt. C without it. (Fig. 2.)

Take any pt. D on the other side of AB;

with centre C and radius CD describe the ⊙ EDF cutting AB in E and F;

and with centres E and F and radii EF and FE respectively, describe ⊙s cutting each other in G; and join CG cutting AB in H.

Then CH ⊥ AB.

For in the two △s CGE, CGF, CE = CF,

CG is common, and GE = EF = GF,

∴ ∠ECG = ∠FCG. (Theor. 13.)

And in the two △s CHE, CHF, CE = CF,

CH is common, and ∠ECH = ∠FCH,

∴ ∠CHE = ∠CHF, and each is a rt. ∠. (Theor. 12.)

IV. Division of a Straight Line into Equal Parts.

PROBLEM 7.

To divide a given straight line into any number of equal parts

Let AB be the given st. line;
suppose it is required to divide it into three equal parts.

From A draw a st. line AE making any ∠ with AB;
take in AE, any three equal parts AC, CD, DE; join EB;
and draw CF, DG ∥ EB.

Then AB is divided into three equal parts in F and G.
For ∵ CF, DG and EB are parallels, and AC=CD=DE,
∴ AF=FG=GB (Theor. 17, Cor. 3).

Cor. 1. If a straight line is drawn through the middle point of one side of a triangle and parallel to the base, it bisects the other side.

And conversely, if a straight line is drawn joining the middle points of two sides of a triangle, it is parallel to the base.

The truth of the former part of this Corollary is evident from the preceding demonstration.

To prove the latter part, let C and F be the middle points of AD, AG; join CF.

Then if FC be not parallel to GD, suppose FC′ ∥ GD.
Then from what precedes,
AC′ = $\frac{1}{2}$ AD = AC, which is impossible unless C and C′ coincide.

Cor. 2. If H be the middle pt. of GD, AFHC, GFCH, and DCFH are parms., and FC = $\frac{1}{2}$ GD, FH = $\frac{1}{2}$ DA, and HC = $\frac{1}{2}$ AG.

V. CONSTRUCTION OF SQUARES, PARALLELOGRAMS, AND TRIANGLES, EQUAL TO OTHER FIGURES.

PROBLEM 8.

To divide a given straight line into two parts such that the rectangle contained by the whole line and one of the parts shall be equal to the square on the other part.

Let AB be the given st. line;
it is required to divide it into two parts such that the rectangle contained by the whole line and one of the parts shall be equal to the square on the other part.

On AB describe the sq. ACDB (Prob. 6 Cor. 2);
bisect AC in E (Prob. 4); join BE; make EF equal to EB;
from F draw FG $\perp$ AF and equal to AF;
and through G draw GHI $\parallel$ AF.
Then H is the pt. of section required.

For $\because$ CA is bisected in E and produced to F,
$\therefore$ rect. CF, FA + sq. on AE = sq. on EF (Theor. 26) = sq. on EB
= sq. on AB + sq. on AE (Theor. 21);
or taking away sq. on AE from both sides,
rect. CF, FA or FCIG = sq. on AB that is ACDB;
and taking away the common part ACIH,
AHGF, that is, sq. on AH = HIDB = rect. BD, BH,
= rect. AB, BH.

NOTE. A straight line thus divided is said to be divided *medially*.

ALGEBRAICAL SOLUTION. This Problem may be algebraically solved thus:—

Let AB $= a$, and one of the parts $= x$.

Then $a(a-x) = x^2$, or $x^2 + ax - a^2 = 0$; $\therefore\ x = \dfrac{-a \pm \sqrt{5}.a}{2}$

$$= \frac{\sqrt{5}}{2}a - \tfrac{1}{2}a$$

(taking only the upper sign).

A comparison of this with the Geometrical solution is instructive.

From the Figure we have $EB^2 = AB^2 + AE^2 = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}$;

$\therefore EB = \frac{\sqrt{5}}{2}a = EF$; and $\therefore$ AF or $AH = EF - EA = \frac{\sqrt{5}}{2}a - \frac{1}{2}a$.

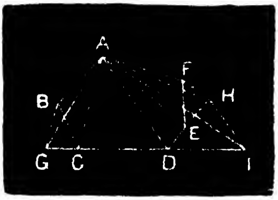
PROBLEM 9.

To construct a parallelogram equal to a given triangle and having an angle equal to a given angle.

Let ABC be the given $\triangle$ and $\angle$ D the given $\angle$;
it is required to construct a parm. equal to $\triangle$ ABC,
and having an $\angle$ equal to $\angle$ D.
Bisect BC in E; make $\angle$ CEF equal to $\angle$ D (Prob. 2);
through A draw AG$\parallel$BC and cutting EF in F;
and through C draw CG$\parallel$EF and meeting AG in G (Prob. 5).
Then CEFG is the parm. required.
For $\because$ BE=EC, $\therefore \triangle ABE = \triangle ACE$ (Theor. 20, Cor. 2),
and $\therefore \triangle$ ABC is double of $\triangle$ ACE.
But parm. CEFG is double of $\triangle$ ACE (Theor. 20, Cor. 1);
$\therefore$ parm. CEFG= $\triangle$ABC; and its $\angle$ CEF= $\angle$ D.

PROBLEM 10.

To construct a triangle equal to a given rectilineal figure.



Let ABCDEF be the given rectilineal figure; it is required to construct a $\triangle$ equal to it, and having A for its vertex and CD produced for its base.

Divide the given figure into $\triangle$s by drawing | s from A.

Draw BG ∥ AC, FH ∥ AE, HI ∥ AD, meeting DC, DE, CD produced in G, H, I; and join AG, AH, AI.

Then AGI is the $\triangle$ required.

For ∵ BG ∥ AC, ∴ $\triangle$ ABC = $\triangle$ AGC.

Again ∵ FH ∥ AE, ∴ $\triangle$ AFE = $\triangle$ AHE;

and ∵ HI ∥ AD, ∴ $\triangle$ AHD = $\triangle$ AID.

Hence $\triangle AGI = \triangle AGC + \triangle ACD + \triangle AID$
$= \triangle ABC + \triangle ACD + \triangle AHD$
$= \triangle ABC + \triangle ACD + \triangle ADE + \triangle AHE$
$= \triangle ABC + \triangle ACD + \triangle ADE + \triangle AFE = ABCDEF.$

Cor. With the help of this and Problem 9, we can make a rectangle equal to a given rectilineal figure.

PROBLEM 11.

To describe a square equal to a given rectilineal figure.

Let R be the given rectilineal figure;
it is required to describe a square equal to R.
Make rectangle ABCD equal to figure R (Prob. 10. Cor.).
Produce AB making BE equal to BC; bisect AE in F;
with centre F and radius FA describe ⊙ AGE;
and produce CB to meet the ⊙ in G.
Then sq. on BG=figure R,
For ∵ AE is bisected in F and cut unequally in B,
∴ rect. AB, BE+sq. on FB=sq. on FE (Theor. 25), =sq. on FG
=sq. on FB+sq. on BG (Theor. 21);
and ∴ taking away sq. on FB from both sides,
rect. AB, BE, that is, rect. AB, BC=sq. on BG.
Hence figure R=rect. AB, BC=sq. on BG.

Cor. If from any point in the circumference of a circle, a perpendicular be drawn to a diameter, the square on the perpendicular is equal to the rectangle under the segments into which it divides the diameter.

Note 1. If AB$=a$, BC or BE$=b$, and BG$=p$, then $p^2=ab$, or $p=\sqrt{ab}$. Thus the number expressing BG is equal to the square root of the product of the numbers representing AB and BC. Hence we may find graphically the square root of a composite number by the following rough method:—

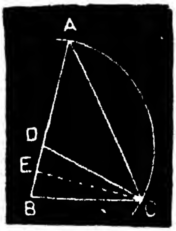
Resolve the number into any two factors; measure with the help of a scale two lines AB, BE which are in the same straight line and which represent, respectively, the two factors; on AE describe a semi-circle; and from B draw BG perpendicular to AE. Then the numerical measure of BG according to the same scale will be the square root required.

Note 2. The solution of this problem (in a somewhat different way) was known to the ancient Hindus. See Journal of the Asiatic Society of Bengal, Vol. 44, p. 245.

VI. Construction of an Isosceles Triangle with Base Angles double of the Vertical.

PROBLEM 12.

To describe an isosceles triangle having each of the angles at the base double of the third angle.



Take any st. line AB; divide it in D so that
AB. BD=AD² (Prob 8); bisect BD in E; draw EC ⊥ BD; with centre D and radius DA describe a ⊙ cutting EC in C; and join CA, CB, CD. Then ABC is the △ required.

For $AC^2 = AD^2 + CD^2 + 2\ AD.\ DE$ (Theor. 23)
$= AD^2 + AD^2 + AD.\ BD$ ($\because BD = 2\ DE$)
$= AD^2 + AB.AD = AB.BD + AB.AD$
$= AB^2$;
$\therefore$ AC = AB, and $\therefore$ △ ABC is isosceles.

Again ∠ B = ∠ BDC ($\because$ △s CEB and CED are congruent)
= ∠ A + ∠ DCA (Theor. 8, Cor. 2)
= ∠ A + ∠ A ($\because$ DA = DC and $\therefore$ ∠ DCA = ∠ A),
that is, ∠ B is double of ∠ A.

Cor. Hence a right angle may be divided into five equal parts

For ∠ A + ∠ B + ∠ BCA = 5 × ∠ A = 2 rt. ∠s,
$\therefore$ ∠ A = $\frac{1}{5}$ of 2 rt. ∠s, and $\frac{1}{2}$ ∠ A = $\frac{1}{5}$ of a rt. ∠.

SECTION IV. EXERCISES.

Introductory Remark. Exercises in Geometry are generally more difficult to work out than those in Algebra or Analytical Geometry, because in the geometrical method of solution there are no fixed rules of procedure such as we have in the algebraical or analytical method; and facility in working out Geometrical examples can be acquired only by practice.

All that could be said by way of general direction would be to tell the student to assume that the deduction, if a theorem, is proved, or, if a problem, is solved; then to examine the figure constructed, to see what known truths or properties, or what known lines or points, the assumption can, step by step, lead to; and lastly, to retrace those steps, proceeding from known truths to those required to be proved, or from given things to those required to be constructed, so as to prove the theorem or solve the problem under consideration.

To quote the words of Proctor in his "First Steps in Geometry," "The average mathematical student requires to learn—not how to solve this or that problem, nor what construction well help him in any particular case; but what are the general methods which he must apply to problems in order to obtain solutions for himself. The mathematical teacher who simply solves the problems brought to him by his pupils, does little to show how such problems are to be treated. He should exhibit to his pupils the train of thought which leads him to apply such and such process to the solution of a problem.................One problem thus dealt with is worth a dozen which are merely solved."

A few Exercises, which are either interesting or which involve important truths, are worked out here; and they are followed by a few more, to be worked out by the student.

Exercises Worked out.

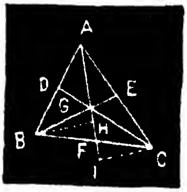
Exercise 1. If from the middle point C and the extremity B of a straight line AB, two parallel straight lines are drawn such that $CD = \frac{1}{2}$ BE, the points A, D, and E are collinear.

For join AE, and suppose AE cuts CD in F.

Then $AF = \frac{1}{2}$ AE (Prob. 7, Cor 1).
and $CF = \frac{1}{2}$ BE (Prob. 7, Cor. 2).

But $CD = \frac{1}{2}$ BE ; ∴ CF = CD, or F and D coincide.

Ex. 2. The three straight lines joining the middle points of the three sides of a triangle with the opposite angles are concurrent.



Let D and E be the middle points of AB AC; let CD, BE be drawn intersecting in G; and let AG be joined and produced to meet BC in F: then if it is shown that BC is bisected in F, the proposition is proved.

Draw BH, CI ⊥ AF

Then ∵ AD = BD, ∴ △ADC = △BDC and △ADG = △BDG, and ∴ taking equals from equals, △AGC = △BGC.

Similarly △BGA = △BGC.

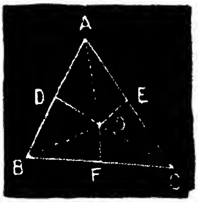
∴ △BGA = △AGC,

and as they are on the same base AG, their altitudes BH and CI are equal (Theor. 20. Note 1).

Hence from △s BFH and CFI, BF = CF (Theor. 14).

Note. The lines CD, BE, AF are called the *medians* of the triangle ABC, and the point G is called its *centroid*.

Ex. 3. The three straight lines bisecting the three sides of triangle at right angles are concurrent.



Let D and E be the middle points of AB, AC and DO, EO perpendiculars to them, and let OF be drawn ⊥ BC; then it it is shown that BF = CF, the proposition is proved.

Now from △s AOD, BOD, AO = BO (Theor. 12). Similarly AO = CO.

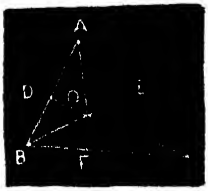
Thus BO = CO, and $BO^2 = CO^2$

Again $BF^2 + OF^2 = BO^2$ (Theor. 21) $= CO^2 = CF^2 + OF^2$;

∴ $BF^2 = CF^2$, and ∴ BF = CF.

Ex. 4. The three straight lines bisecting the three angles of a triangle are concurrent.

Let BO and CO bisect $\angle$s ABC and ACB; then AO shall bisect $\angle$BAC.



Draw OD, OE, OF $\perp$ BA, CA, BC respectively,

Then from $\triangle$s ODB and OFB, OD = OF. (Theor. 14.)

Similarly, OF = OE;

$\therefore$ OE = OD, and $OE^2 = OD^2$.

Now $OD^2 + AD^2 = AO^2$ (Theor. 21)

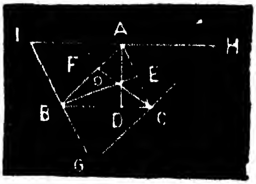
$= OE^2 + AE^2$;

$\therefore AE^2 = AD^2$, and AE = AD;

and $\therefore$ from $\triangle$s OAD, OAE, $\angle$OAD = $\angle$OAE (Theor. 13).

Ex. 5. The three perpendiculars from the three angles of a triangle on the opposite sides are concurrent.

Suppose AD, BE, CF $\perp$ BC, CA, AB respectively: then AD, BE, CF are concurrent.



Through A, B, C draw st. lines $\parallel$ BC, CA, AB forming the $\triangle$GHI.

Then A, B, C are the middle pts. of HI, IG, GH (Prob. 7, Cor. 2), and AD, BE, CF are the perpendiculars bisecting HI, IG, GH, and these perpendiculars, as shewn in Ex. 3, are concurrent.

NOTE. The point of intersection of the perpendiculars on the sides from the opposite angles is called the *orthocentre* of the triangle.

Ex. 6. If the side BC of a right-angled triangle having the right angle B is divided into a number of equal parts BD, DE, EF, FC, and AD, AE, AF are joined, the segments of the angle BAC become smaller and smaller the further they are from AB.

Produce one of the lines AE to G, making EG equal to AE, and join GD.

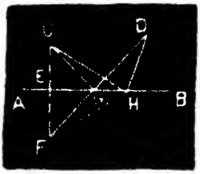
Then from the $\triangle$s AEF, GED, $\angle$EAF = $\angle$EGD, and AF = GD (Theor. 12).

But $\because$ $\angle$ADF > $\angle$AFD, $\therefore$ AF > AD, and $\therefore$ GD > AD.

Hence $\angle$DAE > $\angle$EGD > $\angle$EAF.

Similarly it may be shown that $\angle$EAF > $\angle$FAC.

NOTE. If there be a number of equidistant lamp posts at B, D, E, F, C, to an observer at A they will appear to be more and more close, the further they are from B. The above proposition may help to explain this.



Ex. 7. To find a point in a given straight line at which the straight lines drawn from two given points on the same side of the given line, make equal angles with it.

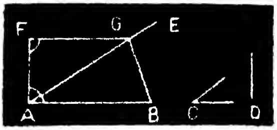
From one of the given points C draw CE $\perp$ AB the given st. line; produce CE making EF equal to EC: join F with D the other given pt., FD cutting AB in G. Then G is the pt. reqd.

For join CG. Then from $\triangle$s CEG and FEG, $\angle$ CGE $=$ $\angle$ FGE (Theor. 12) $=$ $\angle$ DGB (Theor. 3).

If H be any other pt. in AB CH + DH $=$ FH + DH $>$ FD (Theor. 11) $>$ FG + DG $=$ CG + DG; so that the point G is such that the sum of its distances from C and D is a minimum.

Ex. 8. Given the base of a triangle, one of the angles at the base, and the altitude to construct the triangle.

Let AB be the given base,



$\angle$ C the given $\angle$ at the base, and | D the given altitude

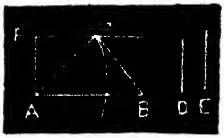
At A in BA make $\angle$ BAE equal to $\angle$ C; draw AF $\perp$ AB and $=$ | D: through F draw FG $\parallel$ AB, and join BG;

Then ABG is evidently the $\triangle$ required.

NOTE. As the required triangle is to have an angle at the base equal to angle C, its vertex must be in AE: and as it is to have its altitude equal to st. line D, its vertex must also be in FG. Hence the vertex must be the intersection of AE and FG, which are respectively the loci of the vertex of a triangle on the given base having the given angle at the base and the given altitude.

Many problems may be solved by the above method of intersection of loci.

Ex. 9. Given the base, the altitude, and one of the other two sides, to construct the triangle.



As in the preceding exercise, draw FG $\parallel$ AB.

Then with centre A and radius equal to the given side E, describe a $\odot$ the $\bigcirc$ of which is the locus of the vertex of a $\triangle$ having the given base and the given side.

And the intersection of the st. line FG with the $\odot$ gives the vertex of the $\triangle$ required.

Ex. 10. If BE and CE bisect $\angle$s ABC and ACD, $\angle E = \frac{1}{2} \angle A$.

For $\angle E + \angle EBC = \angle ECD$ (Theor. 8, Cor. 2)

$$= \tfrac{1}{2} \angle ACD$$
$$= \tfrac{1}{2} \angle A + \tfrac{1}{2} \angle ABC$$
$$= \tfrac{1}{2} \angle A + \angle EBC;$$

$\therefore \quad \angle E = \frac{1}{2} \angle A$.

Ex. 11. In any triangle ABC, if D be the middle point of BC, $AB^2 + AC^2 = 2AD^2 + 2BD^2$

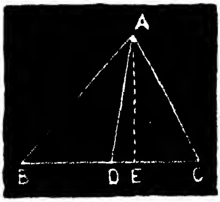
For draw $AE \perp BC$.

Then $AB^2 = AD^2 + BD^2 + 2BD.\ DE$,

and $AC^2 = AD^2 + CD^2 - 2CD.\ DE$, (Theor. 23);

and $BD = CD$;

$\therefore AB^2 + AC^2 = 2AD^2 + 2BD^2$.



Ex. 12. If BC (see last Fig.) is bisected in D and cut unequally in E, $BE^2 + EC^2 = 2BD^2 + 2DE^2$.

For $BE^2 + EC^2 = BC^2 - 2\ BE.\ EC$ (Theor. 24)

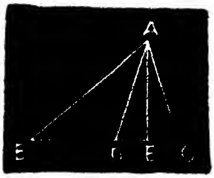
$$= 4BD^2 - 2BE.\ EC \text{ (Theor. 24, Cor. 1)}$$
$$= 2BD^2 + 2BD^2 - 2BE.\ EC$$
$$= 2BD^2 + 2DE^2,$$
$$\because BD^2 = BE.\ EC + DE^2 \text{ (Theor. 25)}.$$

Ex. 13. The difference between the angles at the base of a triangle is equal to twice the angle between the straight line bisecting the vertical angle and the perpendicular from the vertex on the base.

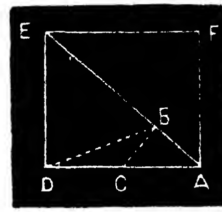
Let AD bisect $\angle BAC$, and draw $AE \perp BC$

Then $\angle C + \angle CAE = \text{a rt. } \angle = \angle B + \angle BAE$.

$\therefore \angle C - \angle B = \angle BAE - \angle CAE$
$= \angle BAD + \angle DAE - \angle CAE$
$= \angle CAD + \angle DAE - \angle CAE$
$= \angle CAE + \angle DAE + \angle DAE - \angle CAE$
$= 2 \times \angle DAE$.



Ex. 14. Given the difference between the diagonal and the side of a square, to construct the square.



Suppose ADEF the sq. required, AB being the given difference between ED and EA.

Draw BC ⊥ BA.

Then ∵ DE = AD,

∴ ∠CAE = ½ rt. ∠ = ∠ACB,

∵ ∠ABC is a rt. ∠;

∴ CB = AB.

Again ∵ ED = EB, ∴ ∠EBD = ∠EDB;
and ∠EDA = a rt ∠ = ∠EBC, ∴ ∠CBD = ∠CDB,
and ∴ DC = CB = BA.

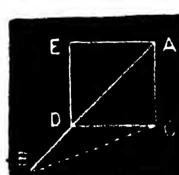
Hence AD, a side of the square may be found thus:—

Draw BC ⊥ AB and equal to AB; join AC, and produce it making CD equal to CB.

NOTE. This mode of regarding the construction as effected and then seeing what follows, will help the solution of problems in most cases.

Ex. 15. Given the sum of the diagonal and the side of a square, to construct the square.

Suppose AB the given sum, and ACDE the required sq.



Then ∠BAC = ½ of a rt. ∠,
and ∠B = ¼ of a rt. ∠.
∵ DC = DB; and
∠ADC = ∠B + ∠DCB = 2 × ∠B.

Hence the solution of the Problem will be as follows:—
At A in BA make ∠BAC equal to ½ of a rt. ∠, at B in AB make ∠ABC equal to ¼ of a rt. ∠, and draw CD ⊥ CA.

EXERCISES TO BE WORKED OUT.

(On Theorems 1—3.)

1. The bisectors of the two adjacent angles which one straight line makes with another are at right angles to each other.

2. The bisectors of each pair of opposite angles which two intersecting straight lines make, are in the same straight line.

3. In the second Figure in Theorem I, if the angle AOC contains 60°, how many degrees are there in the angle BOC and how many in the angle COE ?

(On Theorems 1—7.)

4. If a straight line falls on two intersecting straight lines, the sum of the two interior angles it makes on either side of it, differs from two right angles by the angle between those two lines.

5. If a straight lines falls on two parallel straight lines, the two exterior angles on either side of it are together equal to two right angles.

6. If two straight lines are respectively parallel to two others, and a straight line of the first pair intersects one of the second, the remaining two straight lines must also intersect.

(On Theorems 1—8.)

7. The difference between the angles which the bisector of the vertical angle of a triangle makes with the base, is equal to the difference between the angles at the base.

8. The angle between the bisectors of the angles at the base of a triangle exceeds the vertical angle by the semi-sum of the angles at the base.

9. How many degrees are there in an angle of a regular polygon of 5 sides, and how many in an angle of a regular polygon of 6 sides ?

(On Theorems 1—11.)

10. Prove by the application of Theorems 8 and 9 only that the bisector of the vertical angle of an isosceles triangle is perpendicular to the base.

11. A straight line drawn parallel to the base of an isosceles triangle cuts off an isosceles triangle.

12. A straight line drawn parallel to any side of an equilateral triangle cuts off an equilateral triangle.

13. If from the ends of one side of a triangle two straight lines be drawn to any point within the triangle, their sum shall be less than the sum of the other two sides, but they shall contain a greater angle than that contained by those sides.

14. If two polygons having no re-entrant angle are on the same side of the same base, the perimeter of the outer polygon is greater than that of the inner.

(On Theorems 1—15).

15. The bisector of the vertical angle of an isosceles triangle divides it into two triangles which are equal in every respect.

16. If two straight lines bisect each other, the straight lines joining their extremities form a parallelogram.

17. The bisector of the vertical angle of any triangle divides the base into two parts whereof the one adjacent to the shorter side is less than the other.

18. If from the vertex of a triangle three straight lines be drawn to the base, one perpendicular to the base, the second bisecting the vertical angle, and the third bisecting the base, they are in order of magnitude, the perpendicular being the shortest.

19. If the bisector of the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.

20. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

(On Theorems 1—17.)

21. The diagonals of a rectangle are equal.

22. If the diagonals of a parallelogram are equal, it is a rectangle.

(On Theorems 1—20.)

23. Equal parallelograms on the same side of the same base are between the same parallels.

24. Equal parallelograms on the same base are of the same altitude.

25. If through any point in a diagonal of a parallelogram straight lines are drawn parallel to its sides, the parallelogram will be divided into four parallelograms whereof the two that are not about the diagonal are always equal.

26. The base of a parallelogram is 36 inches, and it contains 9 square feet. Find its altitude.

(On Theorems 1—23.)

27. The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side and the projection of the base on it.

28. If the length of a side of an equilateral triangle is 20 feet, find the length of the perpendicular on it from the opposite angle.

(On Theorems 1—26.)

29. In a right-angled triangle, the square on any of the sides containing the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.

30. The rectangle contained by any two straight lines is equal to the difference of the squares of their semi-sum and semi-difference.

Miscellaneous Exercises.

1. If a straight line is bisected and also cut unequally, the line between the points of section is equal to half the difference between the unequal segments.

2. If an angle is bisected by one straight line and also divided unequally by another, the difference between the two unequal parts is equal to twice the angle between the two dividing lines.

3. In a right-angled isosceles triangle, each of the angles at the base is half a right angle.

4. The perpendicular drawn to the base of a right-angled isosceles triangle from the opposite angle divides it into two equal right-angled isosceles triangles.

5. Trisect a right angle.

6. The diagonals of a square are equal, and bisect each other at right angles.

7. The diagonals of a rhombus bisect each other at right angles.

8. Of all the straight lines that can be drawn to a given straight line from a given point without it, the perpendicular is the least, and of all others, one near the perpendicular is always less than one more remote.

9. In an isosceles triangle, the bisector of the vertical angle bisects the base at right angles; and the perpendicular from the vertex on the base bisects both the base and the vertical angle.

10. If two isosceles triangles stand upon the same base, the straight line joining their vertices bisects the base and also the vertical angles.

11. A quadrilateral with two parallel sides is equal in area to a parallelogram between the same parallels and standing on a base equal to the semi-sum of its parallel sides.

12. The difference between the squares on any two sides of a triangle is equal to the difference between the squares on the segments of the third side made by the perpendicular on it from the opposite angle.

13. The sum of the squares on the sides of any parallelogram is equal to the sum of the squares on its diagonals.

14. The sum of the squares on the sides of any quadrilateral is equal to the sum of the squares on its diagonals, together with four times the squares on the line joining the middle points of the diagonals.

15. If a straight line is cut medially, and from the greater part a part is cut off equal to the less, the greater part is cut medially.

16. Of all rectangles having the same perimeter, the square has the maximum area.

17. Of all rectangles having the same area, the square has the minimum perimeter.

18. Of all triangles on the same base and having the same area, the isosceles triangle has the minimum perimeter.

19. Find the locus of the point the difference between the squares on whose distances from two given points is equal to the square on a given straight line.

20 Find the locus of the point equidistant from two parallel straight lines.

21. Find the locus of the point from which if a perpendicular is drawn to a given straight line, the square on the perpendicular is equal to the rectangle contained by the segments into which that perpendicular divides the straight line.

22. Shew by drawing a correct figure that the triangle whose sides are 3, 4 and 5 inches respectively, is a right-angled triangle.

23. Shew how to trisect the vertical angle of an isosceles triangle each of whose angles at the base is double of the third angle.

24. If two sides of a triangle are 6 and $2\frac{1}{2}$ inches respectively, and they contain a right angle, find the third side and verify your answer by actual measurement.

25. If in the figure in Theorem 21, DF, GI and HE be joined, the sum of the squares of the sides of the hexagon thus formed, shall be equal to eight times the square on the hypotenuse.

26. Each of the medians of a triangle is divided at the point of their ntersection into two parts one of which is double of the other.

27. In any triangle, the sum of the medians is less than the perimeter.

28. If from any point within an equilateral triangle perpendiculars be drawn to the three sides, their sum is equal to one of the medians.

29. Any rectangle is equal to half the rectangle contained by the diagonals of the squares on its adjacent sides.

30. Divide the hypotenuse of a right angled triangle into two parts such that the difference between the squares on them shall be equal to the square on one of the sides.

31. If the sides of a triangle contain linear units equal in number respectively to the sum of the squares, the difference of the squares, and twice the product of any two positive numbers, shew that the triangle is right-angled.

32. If the medians of a triangle ABC interest in O, shew that $AB^2+BC^2+CA^2=3\ (OA^2+OB^2+OC^2)$.

33. If a bamboo measuring 32 cubits and standing upon level ground be broken in one place by the force of the wind, and the tip of it meet the ground at 16 cubits, say at how many cubits from the root it is broken. (Lilavati § 148).

34. A snake's hole is at the foot of a pillar 9 cubits high, and a peacock is perched on its summit. Seeing a snake at a distance of thrice the pillar gliding towards his hole, he pounces obliquely upon him. Say at how many cubits from the snake's hole they meet, both proceeding an equal distance. (Lilavati § 150).

BOOK II.

CIRCLES.

SECTION I. DEFINITIONS.

1. A **chord** of a circle is a straight line joining two points in its circumference.

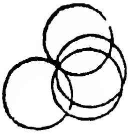
2. A chord produced is called a **secant**.

3. When a secant moves so that the two points of its intersection with the circle continually approach and ultimately coincide, the secant in its ultimate position is called a **tangent** to the circle, and the point at which it meets the circle is called a **point of contact**.

Or, a **tangent** to a circle is a straight line which meets it, and being produced, does not cut it.

4. When one of two intersecting circles moves so that the points of intersection continually approach and ultimately coincide, the two circles in their ultimate position are said to **touch** each other.

Or, circles are said to **touch** one another when they meet but do not cut one another.



5. A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which it divides the circumference; and the other arc is called the **conjugate** arc.

6. An **angle in a segment** is an angle contained by two straight lines drawn from any point in the arc to the extremities of its chord; and it is said to **stand** on the conjugate arc.

7. A **sector** of a circle is the figure contained by two radii and the portion of the circumference between them.

8. A rectilineal figure is said to be **inscribed** in a circle, or the circle is said to be **circumscribed** about it, when the angular points of the figure are on the circumference of the circle.

9. A rectilineal figure is said to be **circumscribed** about a circle, or the circle is said to be **inscribed** in it, when the sides of the figure touch the circle.

Note. In this Book as in Book I, the points, lines, angles, and figures, referred to in any proposition, are supposed to lie in one plane.

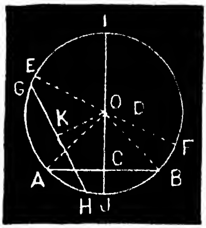
SECTION II. THEOREMS.

I. Chords and Concyclic Points.

HTEOREM I.

I. *A straight line drawn from the centre of a circle to bisect a chord not passing through the centre, is perpendicular to the chord.*

II. *Conversely, the perpendicular from the centre to a chord bisects it.*



I. Let OC be drawn from the centre O to bisect in C the chord AB not passing through O;

then $OC \perp AB$.

Join OA, OB.

Then in the 2 $\triangle$s OCA, OCB,

$AC = BC$, OC is common, and $OA = OB$,

$\therefore \angle OCA = \angle OCB$ (Bk. I. Theor. 13) = a rt. $\angle$.

II. Suppose $OC \perp AB$;

then $AC = BC$.

For $CO^2 + AC^2 = AO^2$ (I. Theor. 21) $= BO^2 = CO^2 + BC^2$;

$\therefore AC^2 = BC^2$, and $\therefore AC = BC$.

Cor. 1. A circle has only one centre which lies in the perpendicular bisecting a chord, and is the middle point of the perpendicular, or is the point of intersection of the perpendiculars bisecting two chords which meet.

For if possible, let O and D be both centres of $\odot$ AHB.

Join OD and produce it to meet the $\odot$ in E and F.

Then $OF = OE = \frac{1}{2}EF$, and $DF = DE = \frac{1}{2}EF$

or $OF = DF$, which is absurd.

As the centre is equidistant from A and B, it lies in ICJ, the perpendicular bisecting AB;

and it must evidently be the middle point O of IJ.

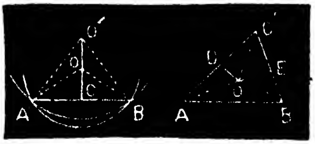
Again as the centre lies in each of the perpendiculars bisecting AB and GH, it must be the intersection of these perpendiculars.

Cor. 2. A diameter of a circle is the *locus of the middle points* of a system of parallel chords to each of which it is perpendicular

THEOREM 2.

I. *Any number of circles can be described passing through two given points.*

II. *Only one circle can be described passing through three given points not in the same straight line.*



I. Let A and B be two given pts.;
then any number of ⊙s can pass through A and B.

For let CO be the perpendicular bisecting AB.

Then ∵ pts. O, O′ etc. in CO are equidistant from A and B,
∴ ⊙s described with centres O, O′ etc. and radii OA, O′ A etc. will pass through A and B.

II. Let A, B, and C′ be any three pts. not collinear;
then only one ⊙ can pass through A, B, C′.

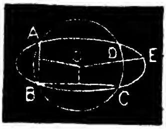
For let DO, EO, be the perpendiculars bisecting AC′, BC′
Then DO and EO must meet.
as AC′ and BC′ are not parallel nor in the same st. line.
Let them meet in O.

Then O is the centre of the ⊙ passing through A, B, and C′.
For, from △s ODA, ODC′, OA = OC′ (I, Theor. 12);
and from △s OEC′, OEB, OC′ = OB;
∴ a ⊙ with centre O and radius OA will pass through A, B, C′.
And no other ⊙ can pass through A, B, and C′,
∵ the st. lines DO and EO
in which the centre of a ⊙ through A, B, and C′ must lie,
can intersect only in one pt. (Axiom 10).

COR. 1. No circle can pass through three points in the same straight line; or in other words, a circle cannot cut a straight line in more points than two.

For, if A, B, and C are pts. in a st. line, the perpendiculars DO, EO′ will be parallel and will not meet, and no ⊙ can be described passing through A, B, and C.

COR. 2. Though any number of circles can have two common points, no two circles can have more common points than two; or, in other words, one circle cannot cut another in more points than two.



For if possible, let ⊙s ABCD, ABCE intersect in A, B, and C.

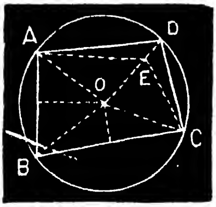
Then A, B, and C are not in a st. line as shewn above; and the centres of the ⊙s is O, the intersection of the perpendiculars bisecting AB, BC.

Draw OA, ODE. Then OA = OD = OE, which is absurd.

THEOREM 3.

I. *If four points are so situated that a circle may be described passing through them, then the opposite angles of the quadrilateral formed by joining them are supplementary.*

II. *Conversely, if the opposite angles of a quadrilateral are supplementary, a circle may be described passing through its four angular points.*



I. Let the the 4 pts. A, B, C, D be such that a ⊙ may pass through them;

then ∠BAD + ∠BCD = 2rt. ∠s = ∠ABC + ∠ADC.

Let O be the centre of the ⊙ ABCD;

join OA, OB, OC, OD.

Then ∵ OA = OB = OC = OD,

∴ ∠OAB = ∠OBA (I, Theor 9), and ∠OAD = ∠ODA;

∴ adding, ∠BAD = ∠OBA + ∠ODA.

Similarly ∠BCD = ∠OBC + ∠ODC.

Hence adding, ∠BAD + ∠BCD = ∠ABC + ∠ADC = 2rt. ∠s (I, Theor. 8, Cor. 3).

II. If ∠BAD + ∠BCD = ∠ABC + ∠ADC = 2rt. ∠s, then a ⊙ may be described passing through A, B, C, and D.

For let the perpendiculars bisecting AB, BC meet in O; then OA = OB = OC. Join OD, and if possible, suppose OD > OA, and OE = OA.

Then a ⊙ passes through A, B, C, E;

and ∴ ∠ABC + ∠AEC = 2rt. ∠s = ∠ABC + ∠ADC (Hyp.);

$\therefore \angle AEC = \angle ADC.$

But $\angle AEO > \angle ADO$, and $\angle CEO > \angle CDO$ (1, Theor. 8, Cor. 2),

$\therefore \angle AEC > \angle ADC.$

Thus $\angle AEC = \angle ADC$ and $> \angle ADC$, which is absurd;

$\therefore$ OD cannot be greater than OA.

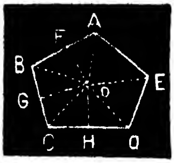
Similarly it may be shown that OD is not less than OA.

Hence OD=OA, and the ⊙ through A, B, C, passes through D.

NOTE. 1. The cases in which O falls outside and on a side of ABCD are left as exercises for the student.

NOTE. 2. Four points lie in the circumference of a circle, that is, are *concyclic*, only when the opposite angles of the quadrilateral formed by joining them are supplementary.

Cor. 1. The angular points of any *regular*, that is, equilateral and equiangular polygon are concyclic.



Take for instance a regular pentagon ABCDE.

Bisect $\angle$s EAB, ABC by AO, BO meeting in O. Join OC.

Then $\angle OAB = OBA$, $\because$ they are halves of equals; $\therefore OB = OA$

Again, in the Δs OBC, OBA, BC=BA, BO is common,

and $\angle OBC = \angle OBA$, $\therefore OC = OA = OB$;

and $\therefore \angle OCB = \angle OBC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle BCD$,

or OC bisects $\angle BCD$.

Similarly it may be shown that OD=OC and bisects $\angle CDE$. And so on.

Hence OA=OB=OC=OD=OE;

and a ⊙ with centre O and radius OA will *circumscribe* the polygon.

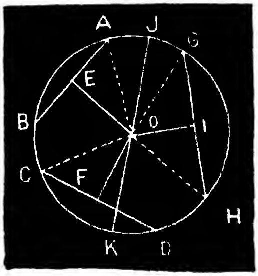
Cor. 2. If from O, OF, OG, OH &c. be drawn $\perp$ the sides of the polygon, F, G, H &c. that is, the feet of the perpendiculars shall be concyclic.

For it may be shown, as in Bk. I. Ex. 4. that OF=OG=OH =&c. Hence a ⊙ with centre O and radius OF will pass through F, G, H &c. It will also *touch* the side of the polygon (see II, Theor. 7) and will be *inscribed* in it.

THEOREM 4.

I. *Equal chords of a circle are equidistant from the centre.*

II. *Conversely, chords of a circle equidistant from the centre are equal.*



I. Let AB, CD be two equal chords of the ⊙ ABCD; then they are equidistant from the centre O, that is, if OE, OF $\perp$ AB, CD, then OE=OF.

Join OA, OC.

Then AB and CD are bisected in E. and F (II, Theor. 1), and AE=CF. ∵ they are halves of the equal chords.

Now $OE^2+AE^2=OA^2=OC^2=OF^2+CF^2$, and $AE^2=CF^2$,

$\therefore OE^2=OF^2$, and OE=OF.

II. Let OE=OF;

then AB=CD.

For $OE^2+AE^2=OA^2=OC^2=OF^2+CF^2$, and $OE^2=OF^2$,

$\therefore AE^2=CF^2$, and $\therefore$ AE=CF,

wut AB=2 AE, and CD=2 CF (II, Theor. 1),

$\therefore$ AB=CD.

Cor. 1. A chord nearer the centre is greater than one more remote.

Suppose OI$\perp$GH and suppose OI<OE; then GH>AB.

For $OI^2+GI^2=OG^2=OA^2=OE^2+AE^2$;

but $OI^2<OE^2$; $\therefore GI^2>AE^2$, and $\therefore$ GI>AE,

or GH>AB.

Cor. 2. The diameter or the chord through the centre is greater than any other chord.

Let JOK be a diameter and GH any other chord. Join OG,OH.

Then JK=OJ+OK=OG+OH>GH (1, Theor. 11).

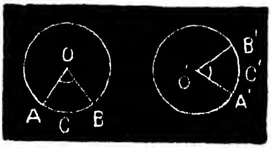
II EQUAL ANGLES AND CHORDS IN EQUAL CIRCLES.

THEOREM 5.

In equal circles or in the same circle,

(i) *if two arcs subtend equal angles at the centre they are equal, and*

(ii) *conversely, if two arcs are equal, they subtend equal angles at the centre.*



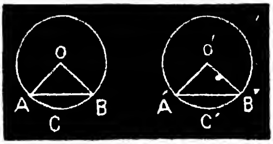
(i) Let arcs ACB, A′C′B′ subtend equal ∠s AOB, A′O′B′ at the centres O, O′ of two equal ⊙s;
then arc ACB = arc A′C′B′.
Apply ⊙ ACB to ⊙ A′C′B′ so that
O may be on O′ and OA on O′A′;
then A shall be on A′ ∵ OA = O′A′ (the ⊙s being equal),
and OB shall be on O′B′, ∵ ∠AOB = ∠A′O′B′,
and arc ACB shall be on arc A′C′B′ ∵ the two ⊙s are equal;
and ∴ arc ACB = arc A′C′B′ (Axiom 9).

(ii) Let arcs ACB, A′C′B′ be equal;
then ∠ AOB = ∠A′O′B′.
Apply ⊙ACB to ⊙ A′C′B′ so that
O may be on O′ and OA on O′A′;
then A shall be on A′ ∵ OA = O′A′ (the ⊙s being equal),
and arc ACB shall be on arc A′O′B′ ∵ the ⊙s are equal,
and B shall be on B′ ∵ arc ACB = arc A′C′B′,
and OB shall be on O′B′, ∵ O and B fall on O′ and B′;
∴ ∠AOB coincides with ∠A′O′B′,
and ∴ ∠AOB = ∠A′O′B′.

If the arcs and angles are in the same circle, in that case O and O′ being coincident, we shall have to apply sector AOB to sector A′O′B′, so that OA may be on OA′; and the rest of the demonstration will be the same as above.

THEOREM 6.

In equal circles or in the same circle,
(i) *if two chords are equal, they cut off equal arcs and*
(ii) *conversely, if two arcs are equal, their chords are equal.*



(i) Let AB, A′B′ be two equal chords of two equal ⊙s;
then arc ACB=arc A′C′B′.
Let O, O′ be the centres of the ⊙s;
join OA, OB, O′A′, O′B′.
Then in the △s AOB, A′O′B′, OA=O′A′, OB=O′B′, AB=A′B′,
∴ ∠AOB=∠A′O′B′ (I, Theor. 13),
and ∴ arc ACB=arc A′C′B′ (II, Theor. 5).

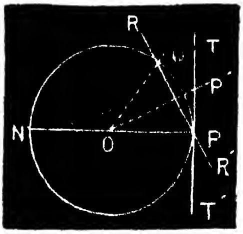
(ii) Let arc ACB=arc A′C′B′,
then chord AB=chord A′B′.
For ∵ arc ACB=arc A′C′B′,
∴ ∠AOB=A′O′B′ (II, Theor. 5).
Now in △s AOB and A′O′B′,
OA=O′A′, OB=O′B′ (∵ the ⊙s are equal),
and ∠AOB=∠A′O′B′.
∴ AB=A′B′ (I, Theor. 12).

If the chords and arcs are of the same circle, evidently, the same demonstration will apply.

III TANGENTS AND TOUCHING CIRCLES.

THEOREM 7.

The tangent to a circle at any point is perpendicular to the diameter through that point.



Let PT be the tangent to the ⊙ NQP at P;
then PT ⊥ diameter NOP.
Draw RQPR′ a secant through P: join OQ;
and produce TP to T′.
Then ∵ OQ=OP, ∴ ∠OPQ=OQP;
and ∠OPR′ + ∠OPQ=2rt, ∠s=∠OQR+∠OQP;
∴ ∠OPR′= ∠OQR.

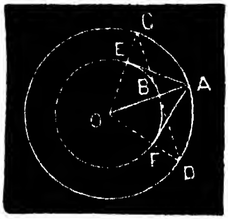
Now if Q continually approaches, and ultimately coincides with, P, the secant ROPR′ continually approaches, and ultimately coincides with, TPT′, and the ∠POQ vanishes, and the equal ∠s OQR, OPR′ become adjacent and coincide with ∠s OPT, OPT′;
∴ ∠ OPT=∠OPT′, and each of them is a rt. ∠.

COR. The tangent TPT′ meets the ⊙ at P but does not cut it.
For take any other pt. P′ in PT and join OP′.
Then ∵ ∠OPT is a rt. ∠, ∴ ∠OPT> ∠OP′P,
and ∴ OP′ > OP (I, Theor. 10), or P′ is outside the ⊙.

NOTE. The truth of this proposition may also be shewn thus: A diameter is the locus of the middle points of a system of parallel chords perpendicular to it (II, Theor. 1, Cor. 2); and as the chord moves further and further from the centre, that is, becomes smaller and smaller, (II Theor. 4. Cor, 1), its extremities approach nearer and nearer and ultimately when the extremities coincide, the chord produced becomes the tangent at the extremity of the diameter.

THEOREM 8.

From any point without a circle, two tangents can be drawn to it, and they are equal, and subtend equal angles at its centre.



Let A be a pt. without the ⊙ EBF;
then two tangents can be drawn from A to the ⊙,
and they shall be equal, and shall subtend equal ∠s at O,
the centre of the ⊙.

Join OA; with centre O and radius OA
describe the ⊙ CAD; from B the pt. at which
OA cuts the ⊙, draw CBD ⊥ OA cutting ⊙ CAD in C and D;
draw OC, OD cutting ⊙ EBF in E and F; and join AE, AF.

Then ∵ in the △s OAE, OCB,
OA=OC, OE=OB, and ∠ AOC is common,
∴ EA=CB, and ∠ OEA=∠ OBC=a rt. ∠
And ∴ AE is a tangent to ⊙ EBF (II, Theor. 7).
Similarly AF is a tangent to ⊙ EBF and=DB.
And CB=DB (II, Theor. 1),
∴ AE=AF.

Again in the △s OAE, OAF,
OE=OF, OA is common, and AE=AF.
∴ ∠ EOA=∠ FOA

THEOREM 9.

If two circles touch, they can have only one point of contact, and the straight line joining their centres passes through the point of contact.

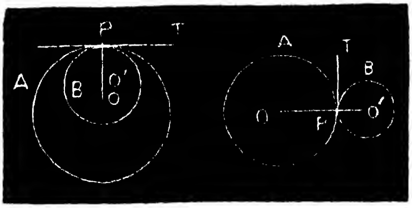


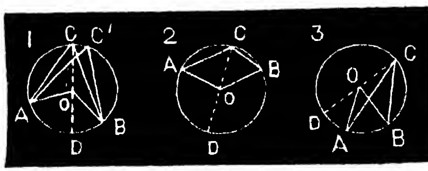
Fig. 1. Fig 2.

Let the two ⊙s PA, PB whose centres are O, O′ touch at P;
then they cannot touch at any other pt.,
and OO′ passes through P.
For ∵ the ⊙s can cut only in two pts. (II, Theor. 2, Cor. 2).
and those two pts. coincide at P, (II, Def. 4),
∴ they cannot touch at any other pt.
And ∵ P is the pt. of ultimate coincidence of the two pts. of intersection of the ⊙s,
they must have a common tangent PT at P, which is the ultimate position of the common secant of the ⊙s through their two approaching points of intersection;
and ∴ OP, O′P must both be ⊥ PT (II, Theor. 7).
Hence ∠OPT = a rt ∠ = ∠O′PT;
and ∴ OP and O′P are either coincident as in Fig. 1,
or in the same st. line (I, Theor. 2) as in Fig. 2.

IV. Angles in Circles.

THEOREM 10.

The angle at the centre of a circle is double of the angle at the circumference standing on the same arc.



Let AOB and ACB be $\angle$s at the centre O and at the $\bigcirc$ of the $\odot$ ACB, standing on the same arc AB;

then $\angle AOB = 2 \times \angle ACB$.

Join CO and produce it to D.

Then $\angle AOD = \angle ACO + \angle OAC$ (1, Theor. 8, Cor. 2)

$= 2 \times \angle ACO$ ($\because \angle OAC = \angle ACO$).

Similarly $\angle BOD = 2 \times \angle BCO$.

Hence adding in Figs. 1 and 2 and subtracting in Fig. 3,

$\angle AOB = 2 \times \angle ACB$.

Cor. 1. Angles ACB and AC′B in the same segment ACC′B are equal.

For each is half of the same $\angle AOB$ at the centre O.

Cor. 2. Conversely, if $\angle ACB = \angle AC'B$, A, C, C′ and B are concyclic.

For if not, let the $\odot$ passing through A, C, B cut AC′ in E (not shown in the figure). Then if we join BE,

$\angle AEB = \angle ACB = \angle AC'B$,

which is impossible (I, Theor. 8, Cor. 2) unless E and C′ coincide.

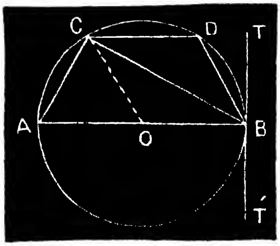
Cor. 3 In equal circles or in the same circle, angles in equal segments are equal.

For these $\angle$s evidently stand on equal arcs, and the angles which these equal arcs subtend at the centres are equal (II, Theor. 5).

Hence the $\angle$s in the equal segments, which are halves of the equal $\angle$s at the centres, are equal.

THEOREM II.

The angle in a semi-circle is a right angle; the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semi-circle is greater than a right angle.



Let ACB be a semi ⊙ AOB being a diameter,
BAC a segment greater than a semi ⊙.
and BDC a segment less than a semi ⊙;
then ∠ ACB = a rt. ∠,
∠ ABC < a rt. ∠,
and ∠ BDC > a rt. ∠.
Join C with O the centre of the ⊙.
Then ∵ OA = OB = OC,
∴ ∠OCA = ∠OAC, and ∠OCB = ∠OBC.
and ∴ ∠OCA + ∠OCB, that is, ∠ACB = OAC + ∠OBC.
But ∠ACB + ∠OAC + ∠OBC = 2 rt. ∠s (I, Theor. 8);
∴ ∠ACB = ½ of 2 rt. ∠s = a rt. ∠.,
And ∠ BAC evidently < a rt. ∠.
Again ∵ ∠BDC + ∠BAC = 2 rt. ∠s. (II, Theor. 3),
and ∠BAC < a rt. ∠,
∴ ∠BDC > a rt. ∠

Cor. If a straight line TBT′ touches the ⊙ACDB, and a chord BC is drawn from the point of contact, the angles it makes with the tangent are equal to the angles in the alternate segments of the circle.

For ∠CBT + ∠ABC = a rt. ∠ (II. Theor. 7)
= ∠BAC + ∠ABC,
∴ ∠CBT = ∠BAC which is in the alternate segment.
Again ∠CBT + ∠CBT′ = 2 rt. ∠s (I. Theor. 1)
= ∠BAC + ∠BDC (II, Theor. 3),
and ∠CBT = ∠BAC,
∴ ∠CBT′ = ∠BDC which is in the alternate segment.

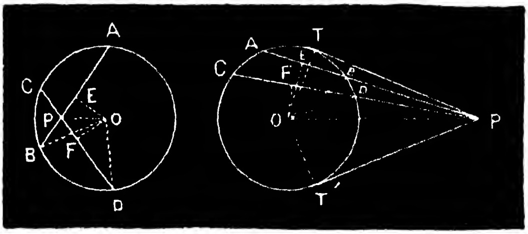
Note. The truth of the above Theorem may be arrived at otherwise thus :—

The $\angle$ABC $= \frac{1}{2}$ $\angle$AOB (II, Theor. 10) $= \frac{1}{2}$ of 2rt. $\angle$s $=$ a rt. $\angle$;
$\angle$CAB $= \frac{1}{2}$ $\angle$COB which is less than $\frac{1}{2}$ of 2rt. $\angle$s or less than a rt. $\angle$;
$\angle$CDB $= \frac{1}{2}$ re-entrant $\angle$COB which is greater than $\frac{1}{2}$ of 2rt, $\angle$s or greater than a rt. $\angle$.

V. INTERSECTING CHORDS AND SECANTS.

THEOREM 12.

If two chords of a circle intersect either within or without the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.



Let the chords AB, CD of the ⊙ ACB intersect in P;
then AP. BP=CP. DP.

From the centre O draw OE, OF⊥AB, CD;
and join OP, OB, OD.

Then AB and CD are bisected in E and F (II, Theor. 1),
and cut unequally in P;

$\therefore$ AP. BP=difference of EB^2 and EP^2 (I, Theor. 25, 26)
=diff. of EB^2+OE^2 and EP^2+OE^2
($\because$ adding OE^2 to both does not alter the diff.)
=diff. of OB^2 and OP^2 (I, Theor. 21)
=diff. of OD^2 and OP^2 ($\because$ OB=OD)
=diff. of OF^2+FD^2 and OF^2+FP^2 (Theor. 21)
=diff. of FD^2 and FP^2 (taking OF^2 from both)
=CP. DP (I, Theor. 25, 26).

Cor. 1. If the chords intersect in P outside the circle, and a tangent PT is drawn to the circle from P, PT^2=AP. BP.

For $PT^2=OP^2-OT^2$ (I. Theor. 21)$=OE^2+EP^2-OB^2$
$=OE^2+EB^2+$AP. BP$-OB^2$ (I, Theor. 26)
$=OB^2+$AP. BP$-OB^2$
=AP. BP.

The same result may be arrived at otherwise thus :—

The tangent PT is the limiting position of the secant PBA when the pts. A and B approach each other and coincide at T, and the segments of the chord become equal to PT; hence $AP.\ BP = TP.\ TP = TP^2$.

Cor. 2. Conversely, if $AP.\ BP = TP^2$, PT is a tangent to the circle.

For draw the tangent PT′ and join OT′.

Then $PT'^2 = AP.\ BP = PT^2$; $\therefore PT' = PT$;

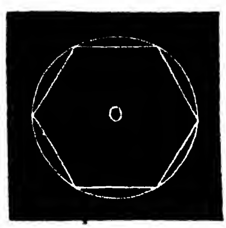
and $\therefore$ in the $\triangle$s PTO, PT′O, $OT = OT'$, $PT = PT'$, and OP is common;

$\therefore \angle PTO = \angle PT'O =$ a rt. $\angle$, and $\therefore$ PT is a tangent.

VI. Polygons Inscribed in and Circumscribed about Circles.

THEOREM 13.

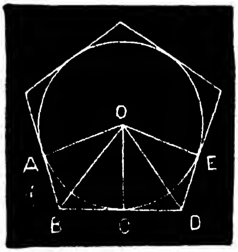
If the circumference of a circle be divided into any number of equal parts, and the consecutive points of division joined by straight lines, regular polygon of the corresponding number of sides will be inscribed in the circle.



For, the polygon will evidently have ,
as may sides as there are divisions of the ○.
The polygon will be equilateral,
∵ its sides being chords of equal arcs are equal (II. Theor. 6).
The polygon will also be equiangular,
∵ each of its ∠s is in a segment composed of
two equal arcs,
and ∴ all its ∠s are equal (II, Theor, 10, Cor. 3).

THEOREM 14.

If the circumference of a circle be divided into any number of equal parts, and tangents to the circle drawn at the points of division, a regular polygon of the corresponding number of sides will be circumscribed about the circle.



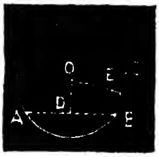
The polygon will evidently have
as many sides as there are divisions of the ○.
The polygon will be equilateral as well as equiangular.
For join the centre O with 3 consecutive pts. of contact A, C, E,
and with 2 intermediate angular pts. of the polygon, B, D.
Then, remembering that tangents from the same pt. are equal,
we have 4 △s whereof the two, OAB, OCB, and the two, OCD, OED,
are equal in all respects (I, Theor. 12), so that,
their ∠s at B and D are each $=\frac{1}{2}$ of the corresponding ∠s of the polygon,
and their ∠s at O are each $=\frac{1}{2}$ of the ∠ on one of the equal arcs.
Again △s OCB, OCD which have their ∠s at O and C equal,
and their side OC common, are equal in all respects,
and ∴ BC=DC, and ∠OBC=∠ODC.
Thus the successive sides of the polygon are doubles of
pairs of equal tangents,
and its successive angles are doubles of equal angles;
that is, the polygon is both equilateral and equiangular.

SECTION III. PROBLEMS.

I. FINDING THE CENTRE OF A CIRCLE.

PROBLEM 1.

To find the centre of a given circle or arc.

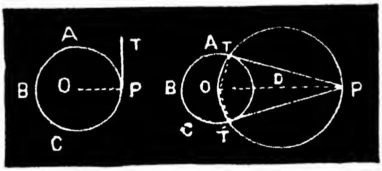


Let ABC be the given ⊙ or arc ;
it is required to find its centre.
Join AB, BC; bisect AB in D and BC in E :
and draw DO, EO ⊥ AB, BC, and meeting in O.
Then O is the centre required.
For the centre being equidistant from A and B,
must be in DO (I, Prob. 6, Cor. 1),
and being equidistant from B and C, it must be in EO ;
∴ the centre is O the intersection of DO and EO.

II. Drawing Tangents to a Circle.

PROBLEM 2.

To draw a tangent to a given circle from a given point.

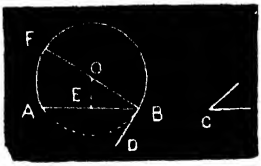


Let ABC be the given ⊙ and P the given pt.;
it is required to draw a tangent to the ⊙ from P.
Find O the centre of the ⊙ and join OP.
If P is on the ○ draw PT ⊥ OP;
then PT is the tangent required (II, Theor. 7).
If P is outside the ⊙, bisect OP in D,
with centre D and radius DO, describe ⊙ OTPT′
cutting the given ⊙ in T and T′; and join PT, PT′.
Then PT and PT′ are the tangents required.
For join OT, OT′.
Then ∵ OTP and OT′P are semi ⊙s,
∴ ∠s OTP and OT′P are rt. ∠s (II, Theor. 11),
and ∴ PT and PT′ are tangents to ⊙ ABC (II, Theor. 7).

III. CONSTRUCTION OF SEGMENTS OF CIRCLES WITH GIVEN CONDITIONS.

PROBLEM 3.

On a given straight line to describe a segment of a circle containing an angle equal to a given angle.



Let it be required to describe on st. line AB
a segment of a ⊙ containing an ∠ equal to ∠C.
At B in AB make ∠ ABD equal to ∠ C;
bisect AB in E; draw BF, EO ⊥ DB, AB, and meeting in O,
and with centre O and radius OA describe ⊙ AFB.
Then AFB is the segment required.
For ∵ O is the centre of ⊙ AFB, and OB ⊥ BD.
∴ BD is a tangent to ⊙ AFB.
And ∵ BD is a tangent and BA is drawn from B,
∠DBA = ∠ in the alternate segment AFB (II, Theor. 11, Cor).
But ∠ DBA = ∠ C;
∴ segment AFB contains an ∠ equal to ∠ C;
and it is described on st. line AB.

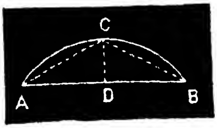
NOTE 1. To cut off from a given circle a segment containing a given angle, we have only to draw a tangent BD, and at B in BD to make an ∠DBA equal to the given ∠; and the segment cut off by BA shall be the segment required.

NOTE 2. The locus of the vertices of triangles on a given base and having a given vertical angle, is the segment of a circle on the given base containing an angle equal to the given vertical angle.

IV. BISECTION OF AN ARC.

PROBLEM 4.

To bisect a given arc.



Let it be required to bisect the arc ACB.

Join AB; bisect AB in D; draw DC $\perp$ AB and meeting the arc in C.

Then the arc is bisected in C.

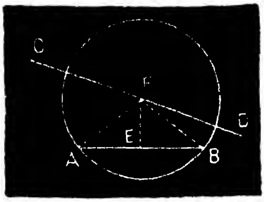
For from $\triangle$s ADC, BDC, AC = BC (I, Theor. 12);

and $\therefore$ arc AC = arc BC (II, Theor. 6).

V. DESCRIBING CIRCLES SATISFYING GIVEN CONDITIONS.

PROBLEM 5.

To describe a circle passing through two given points and having its centre in a given straight line.



Let it be required to describe a ⊙ passing through A and B, and having its centre in | CD.

Join AB; bisect it in E; draw EF ⊥ AB, and meeting CD in F; and join FA, FB.

Then F is the centre and FA the radius of the required ⊙.

For, from △s AEF, BEF, AF = BF (I, Theor. 12);

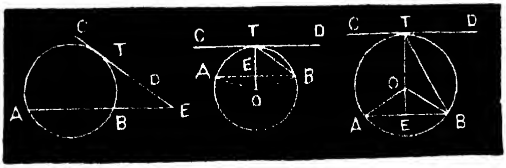
∴ the ⊙ described with centre F and radius FA passes through A and B and has its centre in CD.

NOTE 1. The centre of the required circle must be in EF which bisects AB at right angles, and it must also be in CD; therefore it must be the point F in which EF and CD meet. If CD ∥ EF, the problem is impossible. If CD coincides with EF, the problem is indeterminate, and any point in CD will be the centre of a circle passing through A and B.

NOTE 2. Any number of circles can pass through two points A and B (II. Theor. 2), and therefore a circle passing through A and B can satisfy other conditions. One condition besides passing through two given points is required to be satisfied in this Problem, and another in the next Problem.

PROBLEM 6.

To describe a circle passing through two given points and touching a given straight line.



Let it be required to describe a ⊙ passing through A and B and touching st. line CD.

First, let AB meet CD in E.

Make ET such that $ET^2 = AE.EB$ (I, Prob. 11).

Describe a ⊙ through A, B and T (as shewn in II, Theor. 2).

That ⊙ shall touch ET, ∵ $ET^2 = EA.EB$ (II, Theor. 12, Cor. 2).

Secondly, suppose AB ∥ CD.

Bisect AB in E, draw ET ⊥ CD, make ∠ TBO equal to ∠ BTO.

Then O is the centre and OB the radius of the ⊙ required.

For the ⊙ described with centre O and radius OB will pass through A and touch CD at T,

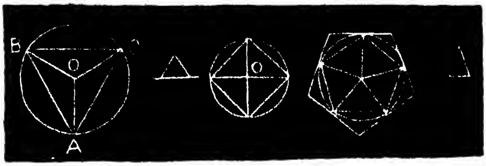
∵ OA = OB = OT, and ∠ OTD is a rt. ∠.

NOTE. If CD meets AB between A and B, the problem is impossible.

VI. INSCRIBING AND DESCRIBING FIGURES IN AND ABOUT CIRCLES.

PROBLEM 7.

In and about a given circle, to inscribe and circumscribe a regular figure of three, four, five, or six sides.



i. To inscribe a regular figure of 3 sides,
the ○, or the 4 rt. ∠s at the centre, must be divided into 3 equal parts so that each ∠ $= \frac{4}{3}$ of a rt. ∠.
Take any equilateral △ (I, Prob. 1); produce one of its sides; draw any radius OA; make ∠s AOB, AOC each equal to an exterior ∠ of the equilateral △.
Then ∠ AOB = ∠ AOC = ∠ BOC, and arc AB = arc AC = arc BC.
Hence ABC is a regular figure of 3 sides (II, Theor. 13).

ii. To inscribe a regular figure of 4 sides,
the ○ must be divided into 4 equal parts.
Draw any diameter, and another ⊥ the former.
These make at the centre 4 equal ∠s standing on equal arcs.
Hence the st. lines joining their extremities will form a regular figure of 4 sides (II, Theor. 13) as required.

iii. To inscribe a regular figure of 5 sides,
the ○ must be divided into 5 equal parts.
Describe an isosceles △ having its ∠s at the base double of the vertical ∠. (I, Prob. 12).
Then an ∠ at the base of this △ $= \frac{4}{5}$ of a rt. ∠.
At O make ∠s each equal to $\frac{4}{5}$ of a rt. ∠.
Thus the ○ will be divided into 5 equal parts, and by joining the pts. of division, the required figure will be inscribed (II, Theor. 13).

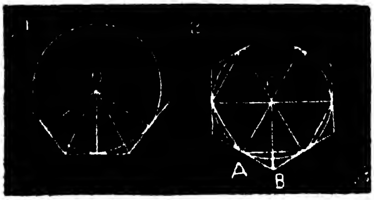
iv. To inscribe a regular figure of 6 sides, bisect the 3 ∠s at the centre in the first figure, and the required points of division of the ○ will be obtained.

v. To circumscribe a regular figure of 3, 4, 5 or 6 sides, divide the ○ into the required number of parts as in the preceding cases, and draw tangents to the ⊙ at the pts. of division, and the required figure will be circumscribed (II, Theor. 14).

VII. Finding the Area of a Circle

PROBLEM 8.

To find the area of a circle.



About the ⊙ circumscribe a regular polygon of n sides, and draw st. lines from the centre to its angular pts. dividing the polygon into n equal Δs.

Let the radius $=r$, the $\bigcirc = c$, and a side of the polygon $=a$.

Then the perimeter of the polygon $= n\,a$,

area of each $\Delta = \frac{1}{2}\,ar$ (I, Theor. 20, Note 2),

and area of the polygon $= \frac{1}{2}\,ar \times n = \frac{1}{2}r \times an$

$= \frac{1}{2}\,r \times$ perimeter of polygon.

Now if n is increased without limit,

the perimeter of the polygon $= c$

area of the polygon $= \frac{1}{2}rc$

and the area of the ⊙ = area of the polygon $= \frac{1}{2}rc$.

From the symmetry of the figure, it will be seen that the relation of c to r must be the same for all circles. (III, Prob. 6, Note 2).

The student will hereafter see that $c = 2\,\pi r$,

and $\therefore$ the area of the ⊙ $= \pi r^2$,

where the Greek letter π (Pi) $= 3{\cdot}14159265\ldots$

He will see that π is an incommensurable quantity the value of which may be found to any degree of approximation. (III, Prob. 6,)

He may easily find that π lies between 3 and $3\frac{1}{2}$.

For $c >$ perimeter of inscribed hexagon, that is $> 6r$,

and $<$ perimeter of circumscribed hexagon, that is < 6OB (Fig. 2).

Now $OA^2 = OB^2 - \frac{1}{4}OB^2 = \frac{3}{4}OB^2$. or $OA = \frac{\sqrt{3}}{2}OB$;

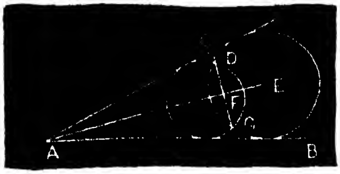
$\therefore\ OB = \frac{2}{\sqrt{3}}OA = \frac{2}{\sqrt{3}}r$, and $6\,OB = 4\sqrt{3} \times r = 6{\cdot}92\ldots r$

$\therefore c < 6{\cdot}92\ldots r$; and $\pi = c \div 2r > 3$ and $< 3\frac{1}{2}$.

SECTION IV. EXERCISES.

Exercises Worked Out.

Ex. 1. To describe a circle passing through a given point and touching two given straight lines.



Let AB, AC be the given | s and D the given pt.

Then ∵ the ⊙ is to touch both AB and AC its centre must be in AE which bisects ∠BAC. (I, Prob. 3. Cor.)

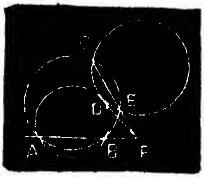
Draw DF ⊥ AE and make FG equal to FD.

Then the ⊙ will pass through G, ∵ it has its centre in AE and passes through D.

Hence the problem is reduced to this, namely, to describe a circle passing through two given points D and G, and touching a given st. line AB or AC (for if it toucnes the one, it must touch the other also) : and this is Problem 6 of this Book. The student should complete the demonstration.

The case in which the st. lines are parallel is left to be solved by the student. It should be noticed that the problem is possible in this case only if the given pt. lies between the given st. lines.

Ex. 2. To describe a circle passing through two given points and touching a given circle.



Let A, B be the given pts., and CDE the given ⊙.

Take any pt. C in CDE, describe a ⊙ passing through A, B, C (as shown in II, Theor. 2) and cutting ⊙ CDE in C and E; produce AB and CE to meet in F; from F draw FD a tangent to ⊙ CDE, and describe the ⊙ ABD through A, B and D. Then ABD shall be the ⊙ required.

For AF. FB = CF. FE = FD² (II, Theor. 12);

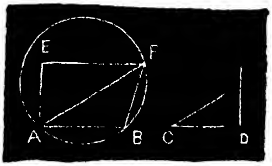
∴ FD touches ⊙ ABD. And ∵ FD also touches ⊙ CDE,

∴ ⊙s ABD and CDE have a common tangent at D;

and ∴ they touch each other at D (see II, Theor. 9).

Ex. 3. Given the base, the vertical angle, and the altitude, to construct the triangle.

On the given base AB describe a segment of a ⊙ containing an ∠ equal to the given ∠ C; draw AE ⊥ AB and equal to given altitude D; draw EF ∥ AB and cutting the segment in F; and join AF, BF. Then AFB is the △ required.

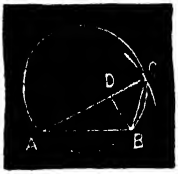


For the vertex of the △ must be in the segment AFB if the △ is to have the given vertical angle, and it must be also in EF if the △ is to have the given altitude.

Hence the vertex must be their intersection F.

Ex. 4. Given the base, the vertical angle, and the sum of the sides, to construct the triangle.

On the given base AB describe a segment of a ⊙ containing an ∠ equal to half the given vertical ∠; with centre A and radius equal to the given sum of the sides describe a ⊙ cutting the segment in C; join AC, BC; at B make ∠ CBD = ∠ C. Then ADB is the △ required.



For it is on the given base. Its vertical

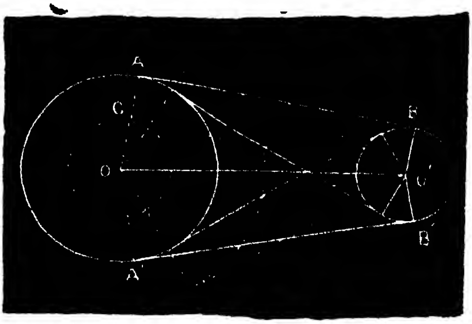
∠ ADB = ∠ DBC + ∠ C = 2 × ∠ C = given vertical ∠. And the sum of its sides AD + DB = AD + DC = AC, the given sum.

Ex. 5. Given the base, the vertical angle and the difference of the sides, to construct the triangle.

On the given base AB describe a segment of a ⊙ ACB containing an ∠ = ∠ DEH, that is, = given vertical ∠ + ½ its supplement; with centre A and radius = given difference describe a ⊙ cutting the segment in C; join AC, BC; produce AC and at B in CB make ∠ CBI = ∠ BCI. Then AIB is the triangle required.

The student should supply the demonstration.

Ex. 6. To draw a common tangent to two given circles.



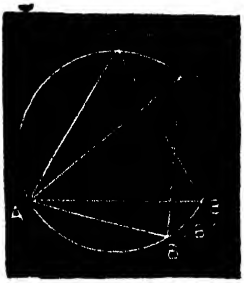
Let O, O′ be the centres of the two ⊙s whereof the former is the greater.

With centre O and radius=difference of the radii of the two ⊙s describe a ⊙ ; from O′ draw O′C a tangent to this ⊙ ; join OC ; produce OC cutting the ⊙ at A ; draw O′B⊥O′C ; and join BA. The | BA is a common tangent to the two ⊙s.

For CA = O′B and CA∥O′B, ∴ AB∥CO′ (I, Th. 17, Cor. 1).
Again ∵ ∠ OCO′ = a rt. ∠ ∴ ∠ A = a rt. ∠ (I, Th. 6).
Similarly ∠ B = a rt. ∠.
And ∴ AB touches the two ⊙s.

The student will see that another tangent A′B′ can be drawn to the two ⊙s on one and the same side of the line joining their centres; and also another pair of common tangents can be drawn each touching the ⊙s on opposite sides of the line joining their centres.

Ex. 7. Of all triangles that may be inscribed in a given circle, the equilateral triangle has the greatest area.



Let ABC be an equilateral $\triangle$ inscribed in the $\odot$ ABC, and A′B′C′ any other $\triangle$ (not shown in the Fig.) inscribed in it.

The $\triangle$ A′B′C′ may be shifted so as to have one of its angular pts. A′ upon A, by measuring from A on both sides arcs = those cut off by A′B′ and A′C′.

Let AB′C′ be the new position of $\triangle$ A′B′C′; and let the $\bigcirc = c$, and arc BB′ $= a$.

Then arc AB′ $= \frac{1}{3}c - a$, and arc ACC′B′ $= \frac{2}{3}c + a$.

Now if C′ be not the middle pt. of arc ACC′B′, it may be easily shewn that $\triangle$AB′C′ will be increased in area by taking C′ to the middle pt. C″, of arc ACC′B′, so that the arc cut off by either of the equal sides of the $\triangle$ thus formed $= \frac{1}{3}c + \frac{1}{2}a$, the arc cut off by the base AB′ being $= \frac{1}{3}c - a$.

The $\triangle$ AB′C″ again will be increased in area by taking B′ to B″ the middle pt. of AB′C″, so that the arc cut off by either of the equal sides $= \frac{1}{3}c - \frac{1}{2^2}$ and the arc cut off by the base $= \frac{1}{3}c + \frac{1}{2}a$.

Proceeding thus, we shall have $\triangle$AB′C′ gradually increased in area, the arcs cut off by either of its equal sides and by its base becoming successively, $\frac{1}{3}c + \frac{1}{2}a$ and $\frac{1}{3}c - a$.

$$\frac{1}{3}c - \frac{1}{2^2}a \text{ and } \frac{1}{3}c + \frac{1}{2}a,$$

$$\frac{1}{3}c + \frac{1}{2^3}a \text{ and } \frac{1}{3}c - \frac{1}{2^2}a,$$

$$\frac{1}{3}c \pm \frac{1}{2^n}a \text{ and } \frac{1}{3}c \mp \frac{1}{2^{n-1}}a, \text{ according as}$$

n is odd or even.

And when n is increased without limit, the arcs approach $\frac{1}{3}c$ in length, and the $\triangle$AB′C′ becomes equilateral, after which there will be no further increase in area.

Ex. 8. In any triangle (ABC) the middle points (D,E,F) of the sides, the feet (G, H, I) of the perpendiculars on them from the opposite angles, and the middle points (J,K,L) of the lines joining the *orthocentre* (O) with the angular points, are concyclic.

We have AF=BF and AJ=OJ,
∴ FJ∥BH (I, Prob. 7, Cor. 1).
Again ∵ BF=AF, and BD=CD,
∴ FD∥AC,
and ∴ ∠DFJ=∠BHA=a rt. ∠.
Similarly ∠DEJ=a rt. ∠.

Hence, F, D, E, J are concyclic and lie on the ○ of a ⊙ whose diameter is JD.

Similarly it may be shewn that K and L are concyclic with D, E, F.

Again ∵ ∠JGD=a rt. ∠,
∴ the semi ⊙ on JD as diameter passes through G, and G is concyclic with D, E, F.

For the same reason, H and I are concyclic with D, E, F. Thus the nine-pts. D, E, F, G, H, I, J, K, and L are concyclic.

NOTE. The circle DEF is called the *Nine Points Circle*.

Exercises to be Worked out.

(On Theorems 1 and 2.)

1. The perpendiculars bisecting the chords of a circle which do not pass through the centre are concurrent.

2. The perpendiculars bisecting any two parallel chords of a circle are in the same straight line.

3. The larger of two circles passing through two given points has its centre in the circumference of the smaller circle. If the diameter of the smaller circle is equal to the distance between the given points, shew that the square on the radius of the larger circle is double of the square on the radius of the smaller circle.

4. If a circle passing through three given points has its centre in the straight line joining two of them, that line subtends a right angle at the third point.

(On Theorems 1 to 4.)

5. If the angular points of a parallelogram are concyclic, the parallelogram is a rectangle.

6. A quadrilateral inscribed in a circle has all its sides equal; shew that its angles are all equal.

7. The middle points of all equal chords of a circle lie on the circumference of a concentric circle; and the difference between the squares on the radii of the two circles is equal to the square on half of one of the equal chords.

8. Of the straight lines that can be drawn from a fixed point within a circle to the circumference, the one passing through the centre is the greatest, and the one which being produced passes through the centre is the least; and of all others, one nearer to the greatest is always greater than one more remote.

(On Theorems 1 to 6.)

9. Of all triangles standing on the chord of an arc and having their vertices in the arc, the triangle that has its vertex at the middle point of the arc is isosceles and has the greatest area.

10. The sides of an equilateral polygon inscribed in a circle, subtend equal angles at the centre.

(On Theorems 1 to 9.)

11. The tangents at the extremities of a diameter are parallel to each other and to the chords bisected by that diameter.

12. The angle between any two tangents to a circle is supplementary to the angle between the radii through the points of contact.

13. The tangents to a circle from any point without it subtend equal angles at the extremities of the diameter through that point.

14. The sums of the opposite sides of a quadrilateral circumscribed about a circle are equal.

(On Theorems 1 to 11.)

15. Of all triangles upon the same base and between the same parallels, the one that is isosceles has the greatest vertical angle.

16. If from any point in the circumference of a circle perpendiculars are drawn to the sides of any inscribed triangle, the feet of the perpendiculars are in a straight line.

(On Theorems 1 to 12.)

17. If a common tangent is drawn to two intersecting circles, the portion of it between the points of contact is bisected by the straight line joining the points of intersection of the circles.

18. If two circles touch each other externally, and two common tangents are drawn to them one of which passes through the point of contact of the circles, shew that the latter bisects the portion of the former lying between the points of contact.

19. Two circles which touch each other externally and whose radii are 2 inches and $4\frac{1}{2}$ inches respectively, have a common tangent drawn to them. Find the length of the portion of it lying between the points of contact.

20. A chord 3 inches long is placed in a circle whose diameter is 5 inches. Find the distance of the chord from the centre.

Miscellaneous Exercises.

1. If two chords of a circle intersect at right angles, the sum of the squares on their segments is equal to the square on the diameter.

2. If a and b be the lengths of the sides of a square and an equilateral triangle inscribed in a circle, shew that $3a^2=2b^2$.

3. In a given square, inscribe the square whose area is the least.

4. In a certain lake, the tip of a bud of lotus seemed a span above the surface of the water. Forced by the wind, it gradually advanced and submerged at the distance of two cubits. Compute the depth of the water, supposing 1 cubit=2 spans. (Lilavati, § 153.)

5. A straight line of given length moves with its extremities on two fixed straight lines at right angles to each other: shew that its middle point moves in a circle.

6. If two circles intersect, the tangents drawn to either of them at their points of intersection meet in the straight line joining their centres.

7. If two circles intersect, the straight line joining their centres bisects at right angles the straight line joining their points of intersection.

8. Parallel chords in a circle intercept equal arcs.

9. If the centres of the *escribed* circles (that is, circles touching one of the sides of a triangle and the other two sides produced) be joined, the orthocentre of the triangle thus formed, is the centre of the circle inscribed in the original triangle.

10. The distance of the point of contact of an escribed circle with either of the produced sides from the vertex of the angle contained by them is equal to the semi-sum of the sides of the triangle; and the distance of the point of contact of the inscribed circle with any side from the vertex of either of the angles adjacent to that side is equal to the difference between the semi-sum of the sides and the side opposite to that angle.

11. If through either of the points of intersection of two equal circles any straight line be drawn meeting them again in two points, these points are equally distant from the other point of intersection.

12. If the vertical angle of a triangle and the inscribed circle remain fixed, the area of the triangle is the least when the triangle is isosceles.

13. If the vertical angle of a triangle and the escribed circle remain fixed, the area of the triangle is the greatest when the triangle is isosceles.

14. Find a point in the circumference of a given circle the sum of the squares of whose distances from two given points may be a maximum.

15. If the straight line joining the centres O and O′ of two circles of radii r and r′ be divided in R so that $OR^2-O'R^2=r^2-r'^2$, and RP is drawn perpendicular to O O′, the tangents drawn from any point P in RP (called the *radical axis* of the two circles) shall be equal.

16. If tangents are drawn to a circle from any point on a fixed straight line, the chord of contact passes through a fixed point (called the *pole* of the fixed straight line which is called the *polar* of the fixed point with respect to the circle).

17. Place a given triangle so that its three sides shall pass through three given points.

18. Place a given triangle so that its three vertices shall lie on three given straight lines.

19. If from any point within a regular polygon of n sides perpendiculars be drawn to the sides, their sum is equal to n times the radius of the inscribed circle.

20. In a circle, prove that an equilateral circumscribed polygon, and an equiangular inscribed polygon, are regular if the number of sides is odd.

BOOK III.

PROPORTIONAL MAGNITUDES AND SIMILAR FIGURES.

SECTION I. DEFINITIONS.

Introductory Remark. Geometrical magnitudes have *position* and *quantity*. In respect of quantity, the only relations of magnitudes, that is, lines, angles, triangles and other figures, which we have hitherto considered, are, those of *equality* and *inequality*. But besides the relations of equality and inequality, there is another important relation, namely, *proportionality*, which magnitudes may bear to one another, and which consists in the equality, not of the *magnitudes themselves*, but of *their relations* in respect of *quantity*.

Thus, if there be two unequal triangles equiangular to one another, any two sides of the one and the two corresponding sides of the other may be all unequal, but the relation of the first mentioned pair of sides to each other in respect of length, is the same as that of the other pair, as will be shewn hereafter. (See Theorem 3 of this Book). So the relation of the side to the diagonal in respect of length is the same in two unequal squares.

This relation in respect of quantity is called **ratio**, and the equality of two ratios is called **proportion**.

DEFINITION 1. **Ratio** is the mutual relation of two magnitudes of the same kind to one another in respect of quantity, the comparison being made by considering what multiple part or parts the first is of the second.

2. If the first of four magnitudes has the same ratio to the second that the third has to the fourth, the four magnitudes are said to be in **proportion** and are called **proportionals**.

3. When three magnitudes are in continued proportion, the first is said to have to the third the **duplicate ratio** of that which it has to the second, and the second is said to be a **mean proportional** between the first and the third.

4. In proportionals, the antecedent terms are said to be **homologous** to one another, and also the consequents to one another.

5. **Similar rectilineal figures** are those that have their several angles equal, and the sides about the equal angles proportional.

NOTE 1. The foregoing definitions require some words of explanation.

From the definition of the term ratio it is clear that if a and b are any two quantities, the ratio of a to b, written, $a : b$, will be represented by the fraction $\frac{a}{b}$ which indicates what multiple, part, or parts a is of b.

If c and d are two other magnitudes such that a, b, c and d are proportionals, then $\frac{a}{b} = \frac{c}{d}$.

And from the above equation, a number of other equations may be easily deduced, as shewn in books on Algebra, which indicate the existence of various important relations among proportional magnitudes. The existence of these relations among proportionals will be assumed in the following pages.

Thus, if $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

And hence if a, b, c, d represent the lengths of four straight lines, then, bearing in mind what is said in Note 2 to I, Theor. 20, we obtain the following theorem :—

If four straight lines are in proportion, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Some of the important relations deducible from the proportion $a : b :: c : d$ are given below for convenience of reference, and the student is required to supply the proofs, by consulting works on Algebra if necessary.

If $\frac{a}{b} = \frac{c}{d}$

then (1) $\frac{b}{a} = \frac{d}{c}$ (*invertendo* or by inversion),

(2) $\frac{a}{c} = \frac{b}{d}$ (*alternando*, or by alternation)

(3) $\frac{a+b}{b} = \frac{c+d}{d}$ (*componendo*, or by composition),

(4) $\frac{a-b}{b} = \frac{c-d}{d}$ (*dividendo*, or by division),

(5) $\frac{a}{a-b} = \frac{c}{c-d}$ (*convertendo*, or by conversion),

(6) $\frac{a+c}{b+d} = \frac{a}{b}$.

The definition of ratio given above may seem to assume that the magnitudes concerned are *commensurable*, that is, capable of being expressed exactly in terms of some common measure or unit.* But geometrical magnitudes are often *incommensurable*. Thus, the diagonal and the side of a square are incommensurable. For if a represents the side, and b the diagonal, then as explained in Note 2 to I, Theor. 21, $b = \sqrt{2}.\ a$, or $\frac{b}{a} = \sqrt{2}$. Now $\sqrt{2}$ cannot be expressed exactly by any number integral or fractional; so that it cannot be said that b is any exact multiple part or parts of a. But as has been shewn in the Note just referred to, the value of $\sqrt{2}$ can be expressed accurately to any required degree of approximation, and a and b can be represented numerically to any required degree of accuracy by adopting a unit of measurement, which is correspondingly small. For if $a = 1$ inch, $\because\ \sqrt{2} = 1.414213......$, taking $\frac{1}{100000}$th of an inch as the unit of measurement, a will be represented by 100000 and b by 141421·3..., and if we stop at the 5th decimal place, the error will be less than $\frac{1}{100000}$th of an inch.

For practical purposes then, all magnitudes may be considered commensurable, all necessary accuracy in the case incommensurable magnitudes being secured by the choice of a unit adequately small; and the definition of ratio will apply equally to all magnitudes whether commensurable or not.

To avoid the difficulty noticed above, arising in the case of incommensurable magnitudes, Euclid has given the following definition, or rather test, of proportionality:—

"The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any equimultiples whatsoever of the first and the third being taken, and any equimultiples of the second and the fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."

This will obviously follow from the Algebraical definition given above. For if $\frac{a}{b} = \frac{c}{d}$, $\frac{ma}{nb} = \frac{mc}{nd}$, and if ma is less than, or equal to, or greater than nb, then mc also is less than, equal to, or greater than nd.

Conversely, the Algebraical definition may also be deduced from Euclid's definition.

For, if a and b are commensurable, suppose $a = nx$ and $b = mx$ where x is a common measure of a and b; then $ma = mnx = nb$, and $\therefore\ mc = nd$, so that $\frac{a}{b} = \frac{n}{m} = \frac{c}{d}$.

If a and b are incommensurable, suppose $b=mx$, and $a>nx$ and $<(n+1)x$, then $ma>mnx$ and $<mnx+mx$, that is, $>nb$ and $<(n+1)b$, and $\therefore mc>nd$ and $<(n+1)d$,

so that $\frac{a}{b}>\frac{n}{m}$ and $<\frac{n+1}{m}$ and also $\frac{c}{d}>\frac{n}{m}$ and $<\frac{n+1}{m}$;

or the difference between $\frac{a}{b}$ and $\frac{c}{d}$ is less than that between $\frac{n+1}{m}$ and $\frac{n}{m}$

that is $<\frac{1}{m}$. And as by making x small, we can make m as large as we please, and as the above relation holds good for all values of m, the difference between $\frac{a}{b}$ and $\frac{c}{d}$ which is $<\frac{1}{m}$, must be less than any assignable quantity, that is, $\frac{a}{b}=\frac{c}{d}$.

NOTE 2. If three quantities, a, b and c, are in continued proportion,

$$\frac{a}{b}=\frac{b}{c}, \text{ and } \therefore \frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}.$$

NOTE 3. In this Book as in the two preceding Books, the points, lines, angles, and figures referred to in any proposition, are supposed to lie in one plane.

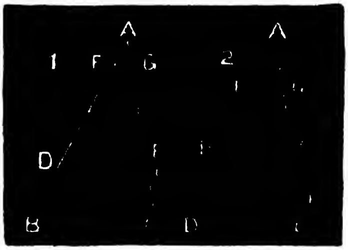
SECTION II. THEOREMS.

I. DIVISION OF SIDES OF A TRIANGLE BY PARALLELS TO THE BASE.

THEOREM 1.

I. *If a straight line is drawn parallel to one side of a triangle, it cuts the other two sides, or those sides produced, proportionally.*

II. *Conversely, if a straight line cuts two sides of a triangle, or those sides produced, proportionally, it is parallel to the third side.*



I. In $\triangle$ ABC, let DE be $\parallel$ BC, cutting in D, E the sides AB, AC in Fig. 1, and AB, AC produced in Fig. 2;

then $\dfrac{AD}{DB}=\dfrac{AE}{EC}$.

Let AF be a common measure of AD, DB,
and let AD $= m$. AF, and DB $= n$. AF.
Divide AD, DB, into m and n equal parts respectively, and through the pts. of section draw st. lines $\parallel$ BC.
Then these lines will divide AE into m and EC into n equal parts (I, Theor. 17, Cor. 3),
so that AE $= m$. AG, and EC $= n$. AG.

Thus $\dfrac{AD}{DB}=\dfrac{m.AF}{n.AF}=\dfrac{m}{n}=\dfrac{m.AG}{n.AG}=\dfrac{AE}{EC}$.

II. To prove the converse,
if DE be not $\parallel$ BC, suppose DE′ $\parallel$ BC,

then $\dfrac{AD}{DB}=\dfrac{AE'}{E'C}=\dfrac{AE}{EC}$ (by hypothesis);

$=\dfrac{AE\pm EC}{EC}$, or $\dfrac{AC}{E'C}=\dfrac{AC}{EC}$;

$\therefore$ E and E′ must coincide, and DE $\parallel$ BC.

NOTE 1. By a method of demonstration similar to that used in this proposition, it may be shown that if a straight line is drawn from the centre of a circle to the arc of any sector, it divides the angle at the centre and the arc on which the angle stands in the same ratio.

For, by taking any small angle which is a common measure of the two angles into which the angle at the centre is divided, and dividing each of the two angles into parts equal to this common measure, and bearing in mind that equal angles at the centre stand on equal arcs, we find that each of the two angles is divided into the same number of equal parts as the arc on which it stands; so that, the two angles into which the angle at the centre is divided are to one another as the arcs on which they stand.

NOTE 2. Similarly, bearing in mind I, Th. 20, Cor. 2, it may be shown that triangles of the same altitude are to one another as their bases.

II. DIVISION OF THE BASE BY THE BISECTOR OF THE VERTICAL ANGLE.

THEOREM 2.

I. *If a straight line bisects the vertical angle of a triangle or its adjacent exterior angle, it cuts the base internally or externally in the ratio of the sides.*

II. *Conversely, if a straight line drawn from the vertex to the base of a triangle cuts the base internally or externally in the ratio of the sides, it bisects the vertical angle or its adjacent exterior angle.*

I. Let AD bisect $\angle$ BAC (in Fig. 1) or its adjacent exterior $\angle$ B'AC (in Fig. 2);

$$\text{then } \frac{BD}{CD} = \frac{BA}{CA}.$$

Draw CE $\parallel$ AD. Then $\angle$ AEC $=$ $\angle$ BAD or $\angle$ B'AD (I, Theor. 6)
$=$ $\angle$ CAD (by hypothesis)
$=$ $\angle$ ACE (I, Theor. 5);
$\therefore$ CA $=$ EA. (I, Theor. 9.)
Again $\because$ AD $\parallel$ CE,

$$\therefore \frac{BD}{CD} = \frac{BA}{EA} \text{ (III, Theor. 1)} = \frac{BA}{CA}.$$

II. Conversely let $\frac{BD}{CD}=\frac{BA}{CA}$;

then $\angle$BAD (in Fig. 1) or $\angle$ B′AD (in Fig. 2)$=$ $\angle$CAD.

Draw CE $\|$ AD.

Then $\frac{BD}{CD}=\frac{BA}{EA}$ (III, Theor. 1) $=\frac{BA}{CA}$ (Hyp.);

$\therefore$ EA $=$ CA, and $\therefore$ $\angle$ACE $=$ $\angle$AEC.

But $\angle$AEC $=$ $\angle$BAD or $\angle$B′AD, and $\angle$ACE $=$ $\angle$CAD (I, Theor. 6 and 5);

$\therefore$ $\angle$BAD or $\angle$B′AD $=$ $\angle$CAD,

that is, AD bisects $\angle$BAC or $\angle$B′AC.

NOTE 1. If BA$=$CA, angle BCA$=$angle CBA, and angle CAB′ (Fig. 2)$=$twice the angle ACB (I, Theor. 8), so that angle CAD$=$ angle ACB, and therefore AD is parallel to BC, and the point D is at an infinite distance. In that case, the proposition is true in this sense that BA : AC being a ratio of equality, BD : CD also is, as it ought to be, one of equality, by reason of BC, the difference between BD and CD being infinitely small compared with BD and CD which are infinitely large.

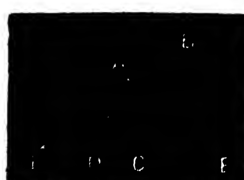
NOTE 2. If AD and AE bisect the angles BAC and B′AC, they cut the straight line BE *harmonically*, that is, in such a manner that the whole line is to one extreme segment as the other extreme segment is to the mean; and the straight lines BD, BC, BE are in *harmonical* progression.

For $\frac{BE}{CE}=\frac{BA}{CA}=\frac{BD}{CD}$,

and $\frac{BE}{BD}=\frac{CE}{CD}$ (alternately).

Again $\frac{BE}{BD}=\frac{CE}{CD}=\frac{BE-BC}{BC-BD}$,

or $\frac{BD}{BE}=\frac{BC-BD}{BE-BC}$.



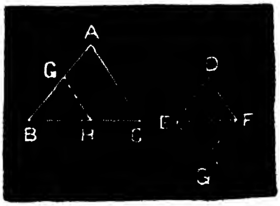
It has been observed that three musical strings of the same material, thickness, and tension, whose lengths are in the same relation as the lines BD, BC and BE, produce notes whose combination is *harmonious* to the ear; and it is from this fact that the name '*harmonical* progression' is derived, and the lines BD, BC, BE are said to be in *harmonical* progression.

III. Similar Triangles.

THEOREM 3.

I. *If two triangles are equiangular, their corresponding sides are proportional, and the triangles are similar.*

II. *Conversely, if two triangles have their sides taken in order proportional, they are equiangular and similar.*



I. Let $\triangle$s ABC and DEF have

$\angle A = \angle D$, $\angle B = \angle DEF$, and $\angle C = \angle DFE$;

then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Apply $\triangle DEF$ to $\triangle ABC$ so that E may be on B, and ED on BA, then EF shall fall on BC, $\because \angle DEF = \angle B$.

Let D and F be at G and H. Join GH.

Then $\because \angle BGH = \angle D = \angle A$, $\therefore$ GH$\parallel$AC (I, Theor. 6);

and $\therefore \frac{BG}{AG} = \frac{BH}{CH}$ (III, Theor. 1); or $\frac{AG}{BG} = \frac{CH}{BH}$ (by inversion),

$\therefore \frac{AB}{BG} = \frac{BC}{BH}$ (by composition). But BG = ED, and BH = EF,

$$\therefore \frac{AB}{DE} = \frac{BC}{EF}.$$

Similarly it may be shewn that $\frac{BC}{EF} = \frac{CA}{FD}$.

II. Conversely, let $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$;

then $\angle A = \angle D$, $\angle B = \angle DEF$, $\angle C = \angle EFD$.

At E and F make $\angle FEG'$ and $\angle EFG' = \angle B$ and $\angle C$.

Then $\angle G' = \angle A$ (I, Theor. 8);

and $\triangle$s ABC, DG'F being equiangular are similar:

$$\text{and } \therefore \frac{AB}{G'E} = \frac{BC}{EF} = \frac{AB}{DE}(\text{Hyp.}); \therefore DE = G'E.$$

Similarly $DF = G'F$. And EF is common to the $\triangle$s $DEF, G'EF$;

$\therefore$ $\triangle$s DEF and $G'EF$ are congruent,

and $\because \angle D = \angle G' = \angle A$, $\angle DEF = \angle G'EF = \angle B$,

and $\angle DFE = \angle G'FE \angle C$.

Hence the two $\triangle$s ABC, DEF are similar.

THEOREM 4.

If two triangles have one angle of the one equal to one angle of the other, and the sides about their equal angles proportional, the triangles are similar.

Let the $\triangle$s ABC and DEF have $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$:

then the $\triangle$s are similar.

Apply $\triangle$ DEF to $\triangle$ ABC so that D may be on A and DE on AB;

then DF shall be on AC, $\because \angle D = \angle A$.

Let E and F be at G and H. Join GH.

Then $\because AG = DE$ and $AH = DF$, and $\frac{AB}{DE} = \frac{AC}{DF}$,

$$\therefore \frac{AB}{AG} = \frac{AC}{AH},$$

$$\text{and } \therefore \frac{BG}{AG} = \frac{CH}{AH} (\text{by division}),$$

and $\therefore$ $GH \parallel BC$ (III, Theor. 1).

Hence $\angle B = \angle AGH = \angle E$, and $\therefore$ $\angle C = \angle F$.

(I, Theor. 8.)

The $\triangle$s ABC and DEF are $\therefore$ equiangular,

and they are $\therefore$ similar (III. Theor. 3).

THEOREM 5.

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in the one proportional to the corresponding sides in the other, then their third angles are either equal or supplementary.

Let the $\triangle$s ABC and DEF have $\angle B = \angle E$, and $\frac{BA}{ED} = \frac{AC}{DF}$;

then $\angle$ C is either equal or supplementary to $\angle$ F.

If $\angle A = \angle EDF$, $\angle C = \angle F$ (I, Theor. 8).

If $\angle$ A be not $= \angle EDF$, make $\angle EDF'$ (Fig. 3) $= \angle A$.

Then $\triangle$s ABC and DEF′ are equiangular and $\therefore$ similar;

$\therefore \frac{BA}{ED} = \frac{AC}{DF'} = \frac{AC}{DF}$ (Hyp.), and $\therefore DF' = DF$, and $\angle C = \angle DF'E$.

Now $\because DF' = DF$, $\therefore$ $\angle F = \angle DF'F$ (1, Theor. 9)
$=$ supplement to $\angle DF'E$
$=$ supplement to $\angle$ C.

NOTE. Theorems 3, 4, and 5 relate to the similarity of two triangles which follows if certain conditions are satisfied. The following are the different cases that arise :—

I. If two triangles are equiangular, they are similar as shown in Theor. 3. There is no case of equality of triangles corresponding to this.

II. If two triangles have their sides taken in order proportional, they are similar as shewn in Theor. 3. The case of equality of triangles corresponding to this is that proved in I, Theor. 13.

III. If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional they are similar as shewn in Theor. 4. The cases of equality of triangles corresponding to this is that proved in I, Theor. 12.

IV. If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in the one proportional to the corresponding sides of the other, the triangles are either similar or have their third angles supplementary as shewn in Theor. 5. The case of equality of triangles corresponding to this is that proved in I, Theor. 15.

IV. SIMILAR POLYGONS AND TRIANGLES.

THEOREM 8.

If a polygon is divided into triangles by straight lines joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.

Let the polygon ABCD be divided into $\triangle$s by | s drawn from O, and let A′B′C′D′E′ be a similar polygon;

then A′B′C′D′E′ can be divided into corresponding similar $\triangle$s.

At A′, B′ make $\angle$s B′A′O′, A′B′O′ = $\angle$s BAO, ABO, respectively, and join O′C′, O′D′ and O′E′.

Then $\triangle$s OAB and O′A′B′ are evidently equiangular;

$\therefore \frac{OB}{O'B'} = \frac{AB}{A'B'}$ (III, Theor. 3) $= \frac{BC}{B'C'} \because$ the polygons are similar.

And $\because \angle ABC = \angle A'B'C'$ and $\angle ABO = A'B'O'$,

$\therefore \angle OBC = \angle O'B'C'$ (Axiom 3).

Hence $\triangle$s OBC and O′B′C′ are similar. (III, Theor. 4.)

Similarly, $\triangle$s OCD, ODE, OEA are similar to $\triangle$s O′C′D′, O′D′E′, O′E′A′.

NOTE. If the given pt. be one of the vertices, A, the proposition may be proved thus:—

Join A′C′ and A′D′.

Then $\triangle$s ABC and A′B′C′ are evidently similar (III, Theor. 4).

And $\triangle$s ACD and A′C′D′, and also $\triangle$s ADE, A′D′E′ may be shown to be similar in the same manner as above.

COR. Hence on a given st. line A′B′ a rectilineal figure may be constructed similar to another figure ABCDE.

For on A′B′ make $\triangle$A′B′C′ equiangular to $\triangle$ABC; on A′C′ make $\triangle$A′C′D′ equiangular to $\triangle$ACD; and on A′D′ make $\triangle$ A′D′E′ equiangular to $\triangle$ ADE.

Then evidently A′B′C′D′E′ is the figure required.

THEOREM 7.

If the straight lines joining a point to the vertices of a given rectilineal figure are divided (all internally or externally) in the same ratio, the points of division are the vertices of a similar and similarly placed rectilineal figure.

Let ABCD be a given rectilineal figure,
and let OA, OB, OC, OD be divided all internally or externally in the same ratio at A′, B′, C′, D′;
then the figures ABCD, and A′B′C′D′ are similar and similarly placed.

For $\because \frac{OA}{OA'} = \frac{OB}{OB'} = \frac{OC}{OC'}$,

$\therefore$ AB ∥ A′B′, and BC ∥ B′C′ (III. Theor. 1),
and $\therefore$ ∠ABC = ∠A′B′C′ (I, Theor. 7, Cor.).
Similarly the other ∠s of the two figures are respectively equal.
Again $\because$ AB ∥ A′B′ and BC ∥ B′C′.
$\therefore$ △s OAB and OA′B′ and also △s OBC and OB′C′ are similar;

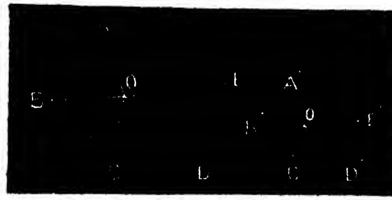
and $\therefore \frac{AB}{A'B'} = \frac{OB}{OB'} = \frac{BC}{B'C'}$.

Similarly the sides of the two figures about their other ∠s are proportional.
Hence the figures are similar.
And they are similarly placed as their corresponding sides are parallel.

Cor. Hence the straight lines joining the vertices of two similar and similarly placed rectilineal figures are concurrent.
For let the st. lines joining AA′ and BB′ meet O.
Join OC, OC′, AC, AC′.
Then △s OCA and OC′A′ may be easily shown to be similar;
$\therefore$ ∠AOC = ∠AOC′ and $\therefore$ OC and OC′ are in the same st. line.

THEOREM 8.

Similar triangles and similar polygons have to one another the same ratio as the squares on their corresponding sides.



i. Let ABC, A′B′C′ be two similar $\triangle$s,
having $\angle$ BAC $= \angle$ B′A′C′, $\angle$B $= \angle$B′, and $\angle$C $= \angle$C′;

$$\text{then } \frac{\triangle ABC}{\triangle A'B'C'} = \frac{BC^2}{B'C'^2}.$$

Draw AD, A′D′ $\perp$ BC, B′C′ respectively.
Then $\triangle$s ABD and A′B′D′ are evidently equiangular and similar;

$$\text{and } \frac{AD}{A'D'} = \frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

$$\text{Now } \frac{\triangle ABC}{\triangle A'B'C'} = \frac{\frac{1}{2}BC.AD}{\frac{1}{2}B'C'.A'D'} \text{ (1, Theor. 20, Note 2.)}$$

$$= \frac{BC}{B'C'} \cdot \frac{AD}{A'D'} = \frac{BC}{B'C'} \cdot \frac{BC}{B'C'} = \frac{BC^2}{B'C'^2}.$$

i. Let ABCDE and A′B′C′D′E′ be two similar polygons;

$$\text{then } \frac{ABCDE}{A'B'C'D'E'} = \frac{AB^2}{A'B'^2}.$$

For the two polygons can be divided into the same number of similar $\triangle$s OAB, OBC, OCD, ODE, OEA, and
O′A′B′, O′B′C′, O′C′D′, O′D′E′, O′E′A′ (III, Theor. 6),

$$\text{and } \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \&c\,;$$

$$\text{and } \frac{\triangle OAB}{\triangle O'A'B'} = \frac{AB^2}{A'B'^2} = \frac{BC^2}{B'C'^2} = \frac{\triangle OBC}{\triangle A'B'C'} = \&c\,;$$

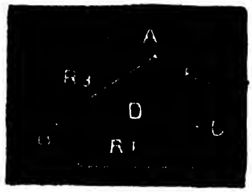
$$\therefore \frac{\text{sum of } \triangle\text{s in ABCDE}}{\text{sum of } \triangle\text{s in A'B'C'D'E'}} = \frac{AB^2}{A'B'^2},$$

$$\text{or } \frac{\text{fig. ABCDE}}{\text{fig. A'B'C'D'E'}} = \frac{AB^2}{A'B'^2}.$$

V. Relation between a Figure on the Hypotenuse and Similar Figures on the other two Sides of a Right-angled Triangle.

THEOREM 9.

In a right-angled triangle, any rectilineal figure described upon the hypotenuse is equal to the sum of the similar and similarly described rectilineal figures upon the sides containing the right angle.



Let ABC be a right angled $\triangle$ having the rt. $\angle$ BAC, and let R_1, R_2, R_3, be similar and similarly described figures on BC, CA, AB;

then $R_1 = R_2 + R_3$.

From A draw $AD \perp BC$.

Then $\triangle$s DBA, DAC are similar to $\triangle$ABC,

$$\text{and} \therefore \frac{CB}{BA} = \frac{BA}{BD} \text{ and } \frac{CB}{CA} = \frac{CA}{CD},$$

and $BA^2 = CB.BD$, and $CA^2 = CB.CD$ (III, Def. 5, Note 1);

$\therefore BA^2 + CA^2 = CB.BD + CB.CD = CB^2$.

$$\text{Now} \frac{R_2}{R_1} = \frac{AC^2}{BC^2} \text{ (III, Theor. 8), and } \frac{R_3}{R_1} = \frac{AB^2}{BC^2};$$

$$\therefore \frac{R_2 + R_3}{R_1} = \frac{AB^2 + AC^2}{BC^2} = \frac{BC^2}{BC^2}; \text{ and } \therefore R_2 + R_3 = R_1.$$

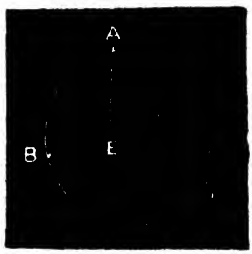
Note. It will be seen from the above proof, that in a right-angled triangle, the three sides are so related that the hypotenuse bears to each side the same ratio which that side bears to the adjacent segment of the hypotenuse cut off by the perpendicular on it from the right angle; so that the square on each side equals the rectangle contained by the hypotenuse and its adjacent segment, and the sum of the squares on the two sides must therefore be equal to the sum of the two rectangles, that is, the square on the hypotenuse. This shews that, Theorem 21 of Book I is deducible from a mere consideration of the relative lengths of the hypotenuse and the other two sides of a right-angled triangle.

This is hinted at in the Vijaganita of Bhaskara, § 146.

VI RELATION BETWEEN THE RECTANGLES CONTAINED BY THE SIDES AND BY THE DIAGONALS OF A QUADRILATERAL INSCRIBED IN A CIRCLE.

THEOREM. 10

The rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to the sum of the rectangles contained by the opposite sides.



Let ABCD be a quadrilateral figure inscribed in the ⊙ ABCD;

then AC. BD = AB. CD + BC. AD.

Make ∠BAE = ∠CAD.

Then ∵ ∠ABD = ∠ACD (II, Theor. 10, Cor. 1),

∴ △s ABE and ACD are equiangular,

and $\therefore \frac{AB}{AC} = \frac{BE}{CD}$ (III, Theor. 3),

and AB. CD = AC. BE.

Again ∠BAE = ∠CAD, ∴ adding ∠EAC to each,

∠BAC = ∠EAD;

and ∠BCA = ∠BDA (II, Theor. 10, Cor. 1);

∴ △s BAC and EAD are equiangular,

and $\therefore \frac{BC}{AC} = \frac{ED}{AD}$ (III, Theor. 3),

and BC. AD = AC. ED.

Hence AB. CD + BC. AD = AC. BE + AC. ED

= AC. BD.

SECTION III. PROBLEMS.

I. DIVISION OF A STRAIGHT LINE IN A GIVEN RATIO.

PROBLEM I.

To divide a given straight line internally and externally in a given ratio.

Let AB be the given st. line and C : D the the given ratio; it is required to divide AB in the ratio of C : D.

From A draw any st. line AE; make AF=C the longer of the two, C and D; make FE=D=FE′; join BE, BE′; and draw FG, FG′∥EB, E′B respectively.

Then G and G′ are the points of division required.

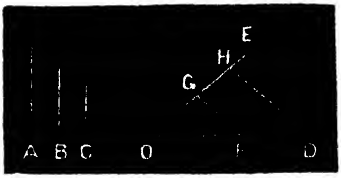
For ∵ FG∥EB, and FG′ ∥ E′B,

$$\therefore \frac{AG}{GB}=\frac{AF}{FE}=\frac{C}{D}=\frac{AF}{FE'}=\frac{AG'}{G'B}\text{(III. Theor. 1).}$$

II. Finding a Third, a Fourth, and a Mean Proportional.

PROBLEM 2.

To find a fourth proportional to three given straight lines.



Let it be required to find a fourth proportional to A, B, C.

Take any two st. lines OD, OE inclined at any $\angle$; make OF = A, FD = B, OG = C; join FG and draw DH $\parallel$ FG.

Then GH is the fourth proportional required.

For $\because$ FG $\parallel$ DH $\therefore \frac{OF}{FD} = \frac{OG}{GH}$, or $\frac{A}{B} = \frac{C}{GH}$.

Cor. Similarly, a third proportional to two given straight lines A and B may be found by making OG = B; and in that case $\frac{A}{B} = \frac{B}{GH}$.

PROBLEM 3.

To find a mean proportional between two given straight lines.

Let it be required to find a mean proportional between A and B.

Take any st. line CD; make CE = A, ED = B; on CD describe the semi-circle CFD; and draw EF $\perp$ CD.

Then EF is the mean proportional required.

For, join CF, DF. Then $\angle$ CFD is a rt. $\angle$ (II, Theor. 11). and $\triangle$s CFE, FDE are equiangular and $\therefore$ similar;

and $\therefore \frac{CE}{EF} = \frac{EF}{ED}$, or $\frac{A}{EF} = \frac{EF}{B}$.

III. Construction of a Figure of a Given Kind and Given Magnitude.

PROBLEM 4.

To describe a rectilineal figure equal to one and similar to another given rectilineal figure.

Let it be required to describe a rectilineal figure equal to the figure A, and similar to the figure BCDE.

Construct the squares FGHI and JKLM equal respectively to A and BCDE (I, Prob. 11): to JK, FG and BC find a fourth proportional NO (III, Prob. 2); and on NO construct the figure NOPQ similar to BCDE (III, Theor. 6. Cor.)

Then NOPQ is the figure required.

For $\because \frac{JK}{FG} = \frac{BC}{NO}$ (by construction),

$$\therefore \frac{JK^2}{FG^2} = \frac{BC^2}{NO^2} = \frac{BCDE}{NOPQ} \text{ (III, Theor. 8)}$$

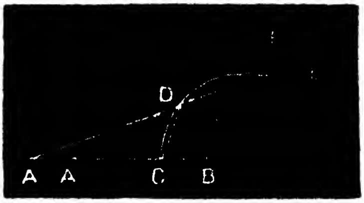
But $BCDE = JK^2$; $\therefore NOPQ = FG^2 =$ figure A; and NOPQ is similar to BCDE.

NOTE. This is the most general Problem relating to the construction of rectilineal figures; and Problem 11 of Book I, the help of which is here taken, is only a particular case of this.

IV. Finding the Locus of the Vertex of a Triangle Satisfying certain Conditions.

PROBLEM 5.

Given the base of a triangle and the ratio of its sides, to find the locus of the vertex.



Divide the given base AB, internally and externally in the given ratio in C and C′, and on CC′ describe the semi-circle CDC′.

This semi-circle is the locus required.

For take any pt. D in the semi-circle and join DA, DB. DC, DC′.

Then if ∠ADB is bisected by DC, its exterior ∠ will be bisected by DC′, ∵ ∠CDC′ being an ∠ in a semi-circle is a rt. ∠,

$$\text{and } \frac{AD}{DB}=\frac{AC}{CB}=\frac{AC'}{C'B}, \text{ the given ratio.}$$

But if possible, let DC be not the bisector of ∠ ADB,
and let ∠ BDC=∠CDA′.
Produce A′D to E′.

Then DC′, will bisect ∠BDE′, ∵ ∠CDC′ is a rt. ∠.

$$\text{Hence } \frac{A'C'}{C'B}=\frac{A'D}{DB}=\frac{A'C}{CB} \text{ or } \frac{A'C'}{A'C}=\frac{C'B}{CB} \text{ (alternately).}$$

$$\text{and } \frac{AC'}{C'B}=\frac{AC}{CB} \text{ or } \frac{AC'}{AC}=\frac{C'B}{CB};$$

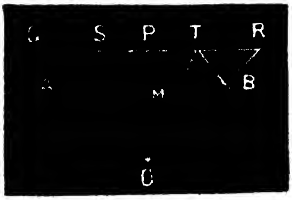
$$\therefore \frac{AC'}{AC}=\frac{A'C'}{A'C} \text{ or } \frac{CC'}{AC}=\frac{CC'}{A'C} \text{ (by division),}$$

and ∴ AC=A′C, that is, A and A′ coincide.

V. Finding the Area of a Circle.

PROBLEM 8.

To find numerically, the approximate area of a circle whose radius is unity, that is, the unit of linear measure ; or, in other words to find approximately the ratio of the circumference to the diameter.



It has been shown in Problem 8 of Book II that the area of a ⊙ of radius $r = \frac{1}{2} cr$ (where c=length of ○)

$$= \pi r^2 \text{ (where } \pi = \frac{c}{2r}\text{).}$$

It has also been shown that $\pi > 3$ and $<$ and $3\frac{1}{7}$.

We now proceed to find to any required degree of approximation.

the numerical value of π, the ratio of the ○ to the diameter, which is also the numerical expression for the area of the ⊙, πr^2 when $r=1$.

For this purpose the following theorem has to be proved, that if I and C are the areas of the regular polygons of n sides inscribed in and circumscribed about the ⊙, and I′ and C′ the areas of like polygons of $2n$ sides,

$$\text{then } I' = \sqrt{I.C}, \text{ and } C' = \frac{2CI'}{C+I'},$$

that is, I′ is the geometric mean between I and C, and C′ is the harmonic mean between I′ and C.

Let AB, QR be the sides of polygons I and C, and AP, ST, those of I′ and C′, as in the Figure.

Then △s OAM, OQP are halves of two of the n △s into which I and C may be respectively divided,

and △s OAP, OST are two of the $2n$ △s into which I′ and C′ may be respectively divided.

Hence $\frac{I}{I'} = \frac{2n\triangle OAM}{2n\triangle OAP} = \frac{\triangle OAM}{\triangle OAP} = \frac{\frac{1}{2}AM \cdot OM}{\frac{1}{2}AM \cdot OP} = \frac{OM}{OP}$

$$= \frac{OA}{OQ} = \frac{\triangle OAP}{\triangle OQP} = \frac{2n\triangle OAP}{2n\triangle OQP} = \frac{I'}{C},$$

or $I'^2 = I.\,C$, or $I' = \sqrt{(I.\,C)}$.

Again, $\frac{SQ}{SP} = \frac{OQ}{OP}$ ($\because$ OS bisects $\angle$ POQ) $= \frac{OQ}{OA}$,

$$\therefore \frac{\triangle OSQ}{\triangle OSP} = \frac{\triangle OQP}{\triangle OAP} = \frac{2n\triangle OQP}{2n\triangle OAP} = \frac{C}{I'};$$

and $\therefore \frac{C+I'}{I'} = \frac{\triangle OSQ + \triangle OSP}{\triangle OSP} = \frac{\triangle OQP}{\triangle OSP} = \frac{4n\triangle OQP}{4n\triangle OSP} = \frac{2C}{C'}$,

or $C' = \frac{2CI'}{C+I'}$.

Now if $r=1$, and $n=4$, that is, if the polygon is a square,

$I = 2$ and $C = 4$,

and $\therefore I' = \sqrt{(I.\,C.)} = \sqrt{8} = 2{\cdot}8284271\ldots$

and $C' = \frac{2CI'}{C+I'} = \frac{8\sqrt{8}}{4+\sqrt{8}} = \frac{16}{2+\sqrt{8}}$

$= 3{\cdot}3137085\ldots$

Proceeding in this way, we have the following table :—

No. of sides.	Area of inscribed polygon.	Area circumscribed polygon.
4	2·00000	4 00000
8	2 82842..	3·31370..
16	3·06146..	3·18259..
32	3·12144..	3·15172..
64	3·13654..	1·14411..
128	3·14033..	3·14222..
256	3·14127..	3·14175..
512	3·14151..	3·14163..
1024	3·14157..	3·14160..
2048	3·14158..	3·14159..
4096	3·14159..	3·14159..

This shews that the areas of the inscribed and circumscribed regular polygons of 4096 sides do not differ down to the 5th decimal place if the radius is 1, that is, their difference is less than $\frac{1}{100000}$th part of the square on the radius or linear unit.

And as the area of the circle lies between these two areas, the difference between that area and either of these two must be still less. Hence if we do not want any higher degree of accuracy, the number 3·14159 may be taken to represent the area of a circle of radius = 1, and also to represent the ratio of the circumference to the diameter.

By proceeding further and further as indicated above, we may secure any degree of accuracy that is desired.

NOTE 1. The above is taken with a slight modification from Legendre's Elements of Geometry, Book IV, Propositions 13 and 14.

NOTE 2. It is assumed above that the ratio of the circumference of a circle to its radius is *constant*, that is, the same for all circles. The truth of this is clear, and may be easily shown thus :—

Inscribe a regular polygon of n sides in each of two circles of radii r and r'; suppose a and a' to be the lengths of the sides of the two polygons; and join the centre of each circle with two of the angular points of the polygon in it. Then the two triangles thus formed are evidently equiangular and therefore similar; and from these triangles we see that a is to a' as r is to r', and therefore na is to na' as r is to r', whatever the value of n may be. Now if n is increased without limit and therefore a and a' diminished without limit, na and na', the perimeters of the polygons, ultimately become equal to the circumferences of the circles. Therefore the circumferences of the circles are to one another as their radii.

SECTION IV. EXERCISES.

1. Triangles between the same parallels are to one another as their bases.

2. Find the locus of the point which divides every straight line passing through it and terminated by two fixed intersecting straight lines in the ratio of its intercepts on those lines.

3. If a quadrilateral figure has two of its sides parallel, the straight line joining the middle points of its other two sides is parallel to the parallel sides.

4. Given the base, the ratio of the other two sides, and the vertical angle, construct the triangle.

5. If from the right angle of a right-angled triangle, a perpendicular is drawn to the hypotenuse, it divides the hypotenuse into segments which are such that each side is a mean proportional between the hypotenuse and the adjacent segment.

6. If in two circles two parallel radii are drrwn, the straight line joining their extremities, produced both ways will cut off similar segments of the circles, standing on chords which are to one another in the ratio of the radii.

7. If a triangle be divided into any two triangles by a straight line drawn from the vertex to the base, the radii of the circles circumscribed about the two triangles shall be in the ratio of the sides.

8. In unequal circles, equal angles at the centres or on the circumference stand on arcs whose chords are proportional to the radii.

9. In unequal circles, similar segments stand upon chords proportional to the radii.

10. Similar triangles are to one another as the squares on the radii of their circumscribing circles.

11. Bisect a triangle by a straight line drawn parallel to one of its sides.

12. The area of a regular hexagon inscribed in a circle is three-fourths of that of a regular hexagon circumscribed about it.

13. If the three sides of a right-angled triangle are in continued proportion, the perpendicular from the right angle on the hypotenuse divides it in extreme and mean ratio, that is, medially.

14. Find the locus of the point which divides in a given ratio straight lines drawn from a given point to a given circle.

15 If two circles touch externally, their common tangent is a mean proportional between their diameters.

16. A circle rolls within another of double its radius. Find the locus of a fixed point in its circumference.

BOOK IV.

PLANES AND SOLIDS.

SECTION I. DEFINITIONS.

Introductory Remark. In the preceding three Books we have considered points, lines, angles, and figures which lie in one plane. We now proceed to consider points, lines, and other geometrical magnitudes lying in different planes.

DEFINITION 1. **A straight line is perpendicular to a plane** when it makes right angles with every straight line meeting it in that plane.

2. **A plane is perpendicular to a plane** when the straight lines drawn in one of the planes perpendicular to the common section of the two planes, are perpendicular to the other plane

3. The **inclination of a straight line to a plane** is the acute angle contained by that straight line and another drawn from the point in which the first mentioned line meets the plane to the point in which a perpendicular to the plane from any point in that line above the plane meets the plane.

4. The **inclination of a plane to a plane**, called also a **dihedral angle**, is the acute angle contained by two straight lines drawn from any the same point in their common section at right angles to it, one in each plane.

5. **Parallel planes** are such as do not meet one another though produced ever so far.

6. **A solid angle** is that which is made by more than two plane angles in different planes meeting in one point.

It is called a **trihedral**, a **tetrahedral**, or a **polyedral** angle, according as it is contained by three, or four, or more than four plane angles.

A solid angle is said to be **convex** when a section of its faces by a plane has no re-entrant angle.

7. **A pyramid** is a polyhedron or solid figure contained by plane figures, of which all the faces except one, called the **base**, meet in a point called the **vertex.**

8. A **prism** is a polyhedron having a pair of parallel faces which are equal and similar rectilineal figures and which are called its ends or **bases**, and having parallelograms for its other faces, called its **side faces**. When the ends are perpendicular to the sides, the prism is called a **right** prism.

9. A **parallelopiped** is a prism with parallelograms for its ends or bases.

10. A **sphere** is a solid figure described by the revolution of a semi-circle about its diameter which remains fixed.

11. A **right cylinder** is a solid figure described by the revolution of a rectangle about one of its sides.

12. A **right cone** is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle.

13. A **tetrahedron** is a solid figure contained by four equal equilateral triangles.

14. A **cube** is a solid figure contained by six equal squares.

15. An **octahedron** is a solid figure contained by eight equal equilateral triangles.

16. A **dodecahedron** is a solid figure contained by twelve equal equilateral and equiangular pentagons.

17. An **icosahedron** is a solid figure contained by twenty equal equilateral triangles.

18. The **projection** of a straight line on a plane is the locus of the feet of the perpendiculars on the plane from the different points of the straight line.

NOTE 1. Some of the foregoing definitions assume the truth of propositions which are hereafter proved. Thus definition 1 assumes what is proved in Theorem 4, namely, that a straight line can be at right angles to all straight lines in a plane which meet it. So definitions 2 and 4 assume what is proved in Theorem 3, namely, that the common section of two planes is a straight line. But the truths assumed are simple and evident.

NOTE 2. It is evident that any number of straight lines may pass through *one* given point, and *two* points determine the position of a straight line.

It is equally evident that any number of planes may pass through *two* given points, that is, through the straight line joining them, and *three* points not in the same straight line, determine the position of a plane.

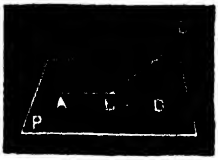
And as a straight line may be made to rotate about any point in it a plane may be made to rotate about any straight line in it.

SECTION II. THEOREMS.

I. Straight Lines in one Plane.

THEOREM I.

One part of a straight line cannot lie in a plane and another part outside the plane.



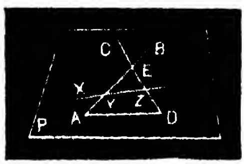
If possible, let AB part of | ABC, be in plane P,
and the part BC without it.
Since | AB is in plane P,
it can be produced to any pt. D in P (I, Post. 2).
Now turn plane P about | AD till it passes through C.
Then | s ABD and ABC lying in one plane
have a common segment, AB, which is impossible (I, Ax. 10).

Note. The truth of this Theorem is evident from the definition of a plane and Axiom 10.

THEOREM 2.

I. Two straight lines which intersect are in one and only one plane.

II. And three straight lines which meet one another but not in the same point, are in one plane.



I. Let AB, CD be two | s intersecting in E;
then they are in one and only one plane.
Let a plane P pass through | AB.
Turn plane P about BA till it passes through pt. C.
Then ∵ C and E are in plane P,
∴ the whole | CED is in plane P (IV, Th. 1).
Moreover | s AB, CD can lie only in plane P.
For, if possible, let them lie also in plane P′.
Take any pt. X in plane P′, and draw XYZ in plane P′
Then ∵ Y and Z are in AB and CD, ∴ they are in plane P;
and ∴ the whole | XYZ is in plane P (IV, Th. 1).
Hence X is in plane P.
Similarly, every other pt. in plane P′ is in plane P,
that is, plane P′ coincides with plane P.

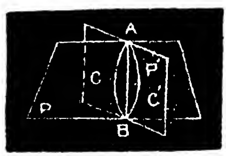
II. Let | s BA, AD, DC meet in A, D, E;
then they are in one plane.
For as shown above, AB and CD are in one plane P.
And ∵ A and D are in this plane, ∴ | AD lies in it.

Note. It will be seen from this proposition that three points not in one straight line, or two intersecting straight lines, determine the position of a plane.

II. Common Section of two Planes.

THEOREM 3.

If two planes intersect, their common section is a straight line.

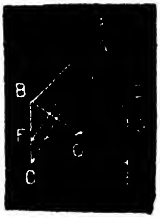


Let the planes, P, P′ intersect in AB;
then AB is a | .
For if not, A and B may be joined by | ACB, AC′B,
drawn in the planes P and P′,
and these lines will enclose a space, which is absurd.
Therefore AB is a st. line.

III. Straight Lines Perpendicular to Planes.

THEOREM 4.

If a straight line is perpendicular to each of two straight lines at their point of intersection, it is perpendicular to every other straight line in their plane which passes through that point, that is, it is perpendicular to their plane.



Let AO be $\perp$ BOE, COD;
then AO $\perp$ any | FOG in plane BOC.
Take any pt. B in OB: make OC, OD, OE=OB:
draw BC, DE cutting FG in F and G; and join B, C,D,E,F,G, to A.
Then $\because$ OB=OE, OC = OD, and $\angle$BOC = $\angle$EOD,
$\therefore$ from Δs BOC, EOD, BC=ED. $\angle$OBC= $\angle$OED (I, Th. 12).
Also $\angle$BOF= $\angle$EOG, and OB=OE;
$\therefore$ from Δs BOF, EOG, BF=EG, and OF=OG (I, Th. 14).
And $\because$ OA bisects at rt. $\angle$s BE and CD,
$\therefore$ AB=AE and AC=AD. Also BC=ED.
Hence Δs, ABC, AED are congruent (I. Th. 13), and
$\angle$ABC= $\angle$AED.
And hence Δs ABF, AEG are congruent (I, Th. 12),
and AF=AG.
Thus in Δs AOF, AOG, OF=OG, OA is common,
and AF=AG:
$\therefore$ $\angle$AOF= $\angle$AOG, or OA $\perp$FG, and $\therefore$ $\perp$*plane BOG.*

THEOREM 5.

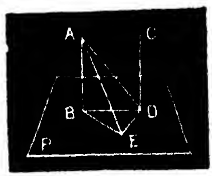
If three or more concurrent straight lines have a common perpendicular at their point of intersection, they are coplanar.

Let OA be $\perp$ OB, OC, OD ;
then OB, OC, OD are coplanar.
For if not, let OD be in a plane
different from that of OB, OC,
and let the plane of OB, OC cut plane AOD in OE.
Then OE is in the plane of OB, OC ;
and $\because$ OA$\perp$OB, OC,
$\therefore$ OA$\perp$OE (IV, Th. 4) ;
and $\therefore$ $\angle$AOE = a rt. $\angle$ = AOD (by hypothesis),
which is absurd,
$\because$ OA, OD, and OE are in one plane, and $\angle$AOD
is part of $\angle$AOE.
Hence OE, OD must coincide, that is, OD must be
coplanar with OB, OC.

THEOREM 6.

I. *If two straight lines are parallel and one of them is perpendicular to a plane, the other is also perpendicular to the same plane.*

II. *Conversely, if two straight lines are perpendicular to the same plane, they are parallel to one another.*

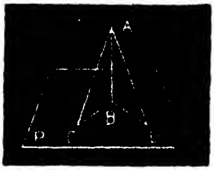


I. Let AB be $\parallel CD$ and $\perp$ plane P;
then $CD \perp$ plane P.
Let AB, CD meet plane P in B and D;
join BD; in plane P draw $DE \perp BD$;
take any pt, E in DE; and join AD, AE, BE.
Then $AE^2 = AB^2 + BE^2$ ($\because \angle ABE$ is a rt. $\angle$ by hypothesis);
$= AB^2 + BD^2 + DE^2$ ($\because \angle BDE$ is a rt. $\angle$ by construction)
$= AD^2 + DE^2$ ($\because \angle ABD$ is a rt. $\angle$ by hypothesis);
and $\therefore DE \perp DA$ (I, Th. 22).
Thus $DE \perp DA$ and DB, and $\therefore \perp$ plane BAD (IV. Th. 4).
Now $CD \parallel AB$, $\therefore CD$ is in the plane BAD;
and $\therefore DE \perp CD$.
Again $\because CD \parallel AB$, and $\angle ABD$ is a rt. $\angle$,
$\therefore \angle CDB$ is also a rt. $\angle$, and $\therefore CD \perp BD$.
Thus $CD \perp BD$ and DE, and $\therefore \perp$ plane P.

II. Let AB, CD be $\perp$ plane P;
then $AB \parallel CD$.
With the same construction as above, we have,
$AE^2 = AB^2 + BE^2 = AB^2 + BD^2 + DE^2 = AD^2 + DE^2$,
$\therefore DE \perp AD$; also $DE \perp BD$ by construction;
and $DE \perp CD$ by hypothesis.
Thus $DE \perp AD$, BD, CD;
$\therefore CD$ is coplanar with AD, BD (IV, Th. 5) and $\therefore$ with AB;
and $\because \angle ABD =$ a rt. $\angle = \angle CDB$, $\therefore AB \parallel CD$ (I, Th. 6).

THEOREM 7.

Of all straight lines drawn from an external point to a plane, the perpendicular is the shortest; and of oblique lines drawn from the same point, those that cut the plane at equal distances from the foot of the perpendicular are equal.



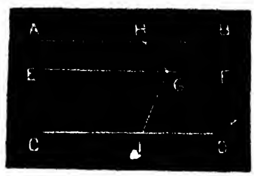
Let AB be $\perp$ plane P, AC any other | ,
and AD a third | such that BD = BC ;
then AB < AC, and AC = AD.
For $\angle$ ABC is a rt. $\angle$, and $\therefore >$ $\angle$ ACB (I, Theor. 8, Cor. 1),
$\therefore$ AC > AB.
And from $\triangle$s ABC, ABD, AC = AD (I, Theor. 12).

Note. This corresponds to I, Theor. 10, Cor.

IV.—PARALLEL STRAIGHT LINES IN SPACE.

THEOREM 8.

Straight lines in space which are parallel to the same straight line, are parallel to one another.



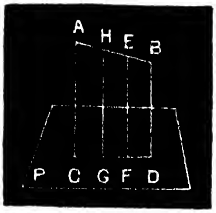
Let | s AB, CD in space be $\parallel$ EF ;
then AB $\parallel$ CD.
From any pt. G in EF, draw
in the plane of AB, EF, the | GH $\perp$ EF,
and in the plane of CD, EF, the | GI $\perp$ EF.
Then $\because$ EF $\perp$ GH, GI, $\therefore$ EF $\perp$ the plane HGI (IV, Theor. 4.)
And $\because$ AB $\parallel$ EF, $\therefore$ AB $\perp$ the plane HGI (IV, Theor. 6.)
For the same reason CD $\perp$ the plane HGI.
Hence AB $\parallel$ CD (IV, Theor. 6).

NOTE. This corresponds to I, Theor. 7.

V. Projection of a Straight Line on a Plane.

THEOREM 9.

The projection of a straight line on a plane is a straight line.

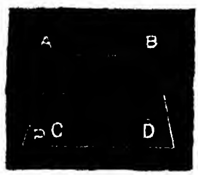


Let AB be a | and P a plane ;
then the projection of AB on P is a | .
Suppose AC, BD $\perp$ plane P, then AC $\parallel$ BD (IV, Theor. 6) ;
and $\therefore$ AC, BD are in the same plane X,
And | AB is in X. $\because$ pts. A and B are in X (I, Def. 7).
Now let CD be the common section of planes X and P.
Then CD is a | (IV, Theor. 3) ; and it is the projection of AB on plane P.
For, take any pt. E in AB and in plane X draw | EF $\perp$ CD ;
then EF $\parallel$ AC and $\therefore$ $\perp$ P, or F is the projection of E on P.
Similarly the projection of every other pt. in AB is in CD.
Again, every pt. in CD is the projection of some pt. in AB.
For take any pt. G in CD and in plane X draw GH $\parallel$ AC.
Then GH $\perp$ P (IV, Th. 6) ; $\therefore$ G is the projection of H on P.

VI. STRAIGHT LINES AND PLANES PARALLEL AND PERPENDICULAR TO ONE ANOTHER.

THEOREM 10.

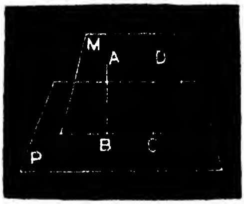
If a straight line outside a given plane is parallel to a straight line in the plane, it is parallel to the plane itself.



Let AB outside plane P be $\parallel$ CD in P;
then AB $\parallel$ plane P.
For $\because$ AB$\parallel$CD $\therefore$ they are in the same plane X;
and if AB produced meet P, its pt of intersection must be in CD the common section of X and P,
which is impossible, $\because$ AB$\parallel$CD.

THEOREM 11.

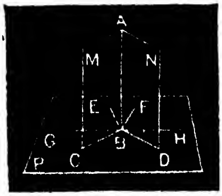
If a straight line is perpendicular to a plane, any plane passing through that line is perpendicular to the first mentioned plane.



Let | AB be ⊥ plane P;
then any plane M through AB is ⊥ P.
For, from any pt. C in the common section BC of P and M
draw CD ⊥ BC.
Then CD ∥ AB; and AB ⊥ plane P;
∴ CD ⊥ plane P. (IV, Theor. 6).
Similarly it may be shewn that
every | in plane M ⊥ BC is ⊥ plane P.
Hence plane M ⊥ plane P.

THEOREM 12.

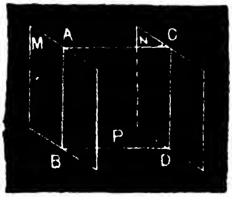
If two intersecting planes are perpendicular to a third plane, their common section is perpendicular to that plane.



Let planes M and N be each ⊥ plane P;
then AB the common section of M and N is ⊥ P.
For if not, from B in plane M draw BE
⊥ BC the common section of M and P,
and in plane N draw BF
⊥ BD the common section of N and P.
Then BE and BF are both ⊥ plane P, which is impossible;
for suppose the plane containing BE, BF cuts P in GH,
then ∠EBG = a rt. ∠ = ∠FBG, which is absurd.

THEOREM 13.

If two parallel planes are cut by a third plane their common sections are parallel.



Let AB, CD be the common sections of the two parallel planes M and N with the plane P;

then AB ∥ CD.

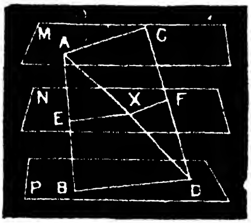
For AB and CD are | s (IV, Theor. 3) in the same plane P, and as they are in the parallel planes M and N, they cannot meet.

Hence AB ∥ CD.

NOTE. A straight line in M can never meet a straight line in the parallel plane N, but they are not parallel lines unless they are in the same plane, that is, are the common sections of M and N with some third plane.

THEOREM 14.

If two straight lines are cut by three parallel planes, they are cut in the same ratio.



Let the | s AB, CD be cut by the parallel planes M, N, P, in A, E, B and C, F, D;

then $\frac{AE}{EB}=\frac{CF}{FD}$.

Draw AD cutting plane N in X; and join EX, FX.

Then ∵ plane N ∥ plane P, and plane ABD cuts them,

∴ EX ∥ BD (IV, Th. 13),

and ∴ $\frac{AE}{EB}=\frac{AX}{XD}$ (III, Th. 1).

Again ∵ plane N ∥ plane M, and plane ADC cuts them,
∴ XF ∥ AC,

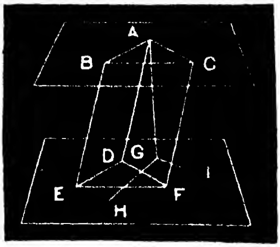
and $\frac{AX}{XD}=\frac{CF}{FD}$. Hence $\frac{AE}{EB}=\frac{CF}{FD}$.

NOTE. This corresponds to III, Theor. 1.

Cor. 1. Parallel planes make equal intercepts on parallel straight lines.

For if AB ∥ CD, then ∵ AC ∥ BD, ∴ ABDC is a parm.,
and ∴ AB=CD.

Cor. 2. If two intersecting straight lines in space are parallel to two others, (i) the first two and the other two contain equal angles, and (ii) they are in parallel planes.

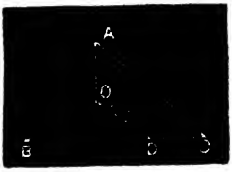


Let AB, AC be respectively ∥ DE, DF.
Then (i) ∠ BAC= ∠ EDF.
For make AB, AC = DE, DF, and join AD, BE, CF, BC, EF.
Then ∵ AB=and ∥ DE, ∴ BE = and ∥ AD;
and ∵ AC=and ∥ DF, ∴ AD = and ∥ CF.
And hence ∵ BE=and ∥ CF, ∴ BC = and ∥ EF.
Thus from △s BAC, EDF, ∠ BAC = ∠ EDF. (I, Th. 13).
And (ii) plane ABC ∥ plane DEF.
For let AG be ⊥ plane DEF. Draw GH, GI ∥ AB, AC.
Then ∵ AG ⊥ plane DEF, ∠s AGH, AGI are rt. ∠s;
and ∵ AB, AC ∥ GH, GI, ∠s GAB, GAC are rt. ∠s (I, Th. 6).
Hence AG ⊥ planes ABC and DEF.
and ∴ plane ABC ∥ plane DEF.
For if they meet, | s drawn from any pt. in their common section
to A and G will make rt. ∠s with AG,
and thus two ∠s of a △ will be two rt. ∠s,
which is impossible (I, Th. 8, Cor. 1).

VII TRIHEDRAL ANGLES.

THEOREM 15.

In a trihedral angle, the sum of any two face angles is greater than the third.



Let the trihedral ∠ at O be contained by the three ∠s AOB, BOC, COA;

then any two of these, ∠AOB + ∠AOC > ∠BOC.

If ∠BOC < or = ∠AOB, the proposition is evidently true.

Suppose ∠BOC > ∠AOB.

At O in BO make ∠BOD = ∠BOA;

take any pts. B, C in OB, OC;

draw BC cutting OD in D; make OA = OD; and join AB, AC.

Then from △s OBA, OBD, BA = BD (I, Th 12).

But BA + AC > BC (I, Th. 11), that is, > BD + DC,

∴ AC > DC.

Thus in △s AOC, DOC we have, OA = OD, OC common,

and AC > DC; ∴ ∠AOC > ∠DOC (I, Th. 16).

Hence ∠AOB + ∠AOC > ∠BOD + ∠DOC,

that is, > ∠BOC.

THEOREM 16.

If two trihedral angles have the face angles of the one equal to those of the other, each to each, the trihedral angles are equal.

Let the trihedral $\angle$s at O and o have the face $\angle$s of the one respectively equal to those of the other; then the trihedral angles are equal to one another.

Take any pt A in OA; make $oa = OA$;
in the planes OAB, OAC draw AB, AC $\perp$ OA;
in the planes oab, oac draw AB, $ac \perp oa$; and join BC, bc,

Then in $\triangle$s OAB, oab, $\because$ $\angle AOB = \angle aob$ (by hypothesis),
$\angle OAB = \angle oab$ (each being a rt. $\angle$)
$OA = oa$ (by construction),
$\therefore$ $OB = ob$, $AB = ab$ (I, Th. 14).

Similarly from $\triangle$s OAC, oac, $OC = oc$, $AC = ac$.
And $\because$ $\angle BOC = \angle boc$ (by hypothesis).
$\therefore$ from $\triangle$s OBC, obc, $BC = bc$ (I, Th. 12).

Hence from $\triangle$s ABC, abc, $\angle BAC = \angle bac$ (I, Th. 13);
that is, the inclination of face OAB to face OAC
= the inclination of face oab to face oac (IV, Def. 4).

Similarly it may be shewn that
the inclinations of the faces OAB, OAC to OBC are respectively equal to the inclinations of the corresponding faces of the solid angle at o.

Hence, if the face angles are *similarly situated*,
the equality of the solid angles may be shewn by *superposition*.

If the face angles are *not similarly situated*,
the solid angles may be placed so that two of their corresponding faces may coincide, and the solid angles shall lie *symmetrically* on opposite sides of the plane of coincidence, and must be equal by *symmetry*.

NOTE. 1. The cases in which the perpediculars on any of the edges of the solid angles do not meet their other edges are left as exercises for the student.

NOTE 2. It should be observed that equal magnitudes do not always coincide, and that the converse of Axiom 9 is not always true.

VIII. Convex Solid Angles.

THEOREM 17.

The sum of the face angles of any convex solid angle is less than four right angles.

Let the solid $\angle$ at O be contained by
the plane $\angle$s AOB, BOC, COD, DOE, EOA;
then their sum <4 rt. $\angle$s.

Let a plane cut the planes of the face $\angle$s
in AB, BC, CD, DE, EA; suppose any pt. X taken within
the figure ABCDE; and from X suppose XA, XB,
XC, XD, XE drawn.

Then the sum of the face $\angle$s at O
$+$ the sum of the base $\angle$s of $\triangle$s OAB, OBC, etc.
$=$ all the $\angle$s of $\triangle$s OAB, OBC etc.
$=$ twice as many rt. $\angle$s as there are $\triangle$s
$=$ all the $\angle$s of the $\triangle$s XAB, XBC etc.
($\because$ the number of $\triangle$s in each case is the same)
$=$ all the $\angle$s of the fig. ABCDE $+$ the $\angle$s at X
$=$ all the $\angle$s of the fig. ABCDE $+4$ rt. $\angle$s.

Now each of the solid $\angle$s at A, B, C, D, E, is a trihedral $\angle$
contained by two of the base $\angle$s of the $\triangle$s OAB, OBC etc.
and one of the angles of the figure ABCDE;
and as any two of these $\angle$s in each case $>$ the third (IV, Th. 15),
$\therefore$ the sum of the base $\angle$s of the $\triangle$s OAB, OBC etc.
$>$ the $\angle$s of the figure ABCDE;
and $\therefore$ the sum of the face $\angle$s at O <4 rt. $\angle$s.

Cor. There cannot be more than five *regular* solids, that is, solids whose faces are equal regular rectilineal figures.

For every solid angle of a regular solid must satisfy two conditions :—

(i) It must be contained by at least 3 plane $\angle$s.

(ii) The sum of the plane $\angle$s by which it is contained must be $<$ 4 rt. $\angle$s.

Regular figures of more than 5 sides are $\therefore$ excluded, $\because$ 3 $\angle$s of such a figure are not $<$ 4 rt. $\angle$s.

Again more than 5 $\angle$s of a regular figure of 3 sides, and more than 3 $\angle$s of one of 4 or 5 sides are excluded by condition (ii).

Thus the only regular solids possible are those whose solid angles are contained

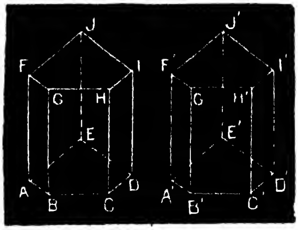
(1) by 3 $\angle$s of equilateral $\triangle$s (as in the tetrahedron),
(2) by 4 $\angle$s of ,, ,, (,, octahedron),
(3) by 5 $\angle$s of ,, ,, (,, icosahedron),
(4) by 3 $\angle$s of squares (,, cube),
(5) by 3 $\angle$s of regular pentagons (,, dodecahedron).

NOTE. To construct the regular solids by cutting and folding card-board or stiff paper see Problem 3 at page 160.

IX. Volumes of Prisms, Parallelopipeds, and Pyramids.

THEOREM 18.

Right prisms of equal altitudes and on equal and similar bases are equal.



Let ABCDE—FGHIJ, A'B'C'D'E'—F'G'H'I'J'
be two right prisms of equal altitudes AF, A'F'
and on equal and similar bases ABCDE and A'B'C'D'E';
then they are equal.

For if the prism A'B'C'D'E'—F'G'H'I'J' be applied to
the prism ABCDE—FGHIJ so that
the pt. A' may be on A, and | A'B' on AB,
then B' shall be on B ∵ A'B'=AB,
| B'C' shall be on | BC, ∵ ∠A'B'C'=∠ABC,
C' shall be on C, ∵ B'C'=BC, and so on;
that is, figure A'B'C'D'E' shall coincide with ABCDE.
Again A'F' shall be on AF, ∵ each is ⊥ its base,
and F' shall be on F ∵ A'F'=AF.
And for the same reason, G',H,'I',J', shall be on G,H,I,J,
and the entire prism A'B'C'D'E'—F'G'H'I'J' shall coincide
with prism ABCDE—FGHIJ.
Hence the prisms are equal.

NOTE. The prisms are supposed to be hollow and their faces to be imaginary planes perfectly penetrable.

COR. 1. Right prisms of equal altitudes and on equal bases which are parallelograms are equal.

For each base may be easily transformed into its equivalent rectangle having a common side, and the parallelogram may be cut into parts and made to coincide with the corresponding

rectangle (I. Th. 18. Note 1); and the prism on each base may be cut into parts in the same manner by planes perpendicular to the base, and made to coincide with a right prism of the same altitude on the corresponding rectangular base. Thus each prism will be transformed into its equivalent right prism of the same altitude on a rectangular base of equal area. Again, the two rectangular bases may, by the proper selection of a common linear unit, be each divided into the same number of equal squares, and the two prisms on the rectangular bases may each be divided into the same number of prisms of the same altitude on bases which are squares on the linear unit, and all these prisms will be evidently equal. Hence the original prisms must also be equal.

Cor. 2. Hence, right prisms of equal altitudes and on equal bases which are triangles, are equal.

For they are evidently halves of prisms of equal altitudes on bases which are parallelograms of equal area, and these last mentioned prisms are, by the preceding Corollary, equal.

Cor. 3. And hence generally, right prisms of equal altitudes and on equal bases which are any rectilineal figures, are equal.

For each base may be cut into trianges as in Bk. I, Prob. 10, and the prism on each base may be cut into as many triangular prisms by planes $\perp$ the base and passing through the lines of division of the base; and each of these triangular prisms being, by Cor. 2, equal in volume to a triangular prism of equal altitude on a base equivalent to its own, it is clear that each original prism is equal to a prism of equal altitude on an equivalent triangular base into which its original base may be transformed by the help of the above mentioned Problem.

Thus the original prisms may be transformed into their equivalent triangular prisms of equal altitudes and bases; and these last mentioned prisms are equal by Cor. 2.

Cor. 4. If two prisms have their bases and side faces respectively congruent, they are equal.

For the face angles containing the solid angles of the one, which are all trihedral, being respectively equal to those in the other, the prisms have their corresponding solid angles equal (IV, Th. 16). They have also their corresponding edges equal. Hence the prisms must be equal.

THEOREM 19.

Parallelopipeds having the same base and equal altitudes are equal.

Fig. 1.

Let ABCD—EFGH and ABCD—E′F′G′H′
be two parallelopipeds on the same base ABCD,
and having the same altitude;
then they are equal.

First, let the edges EF, E′F′ be in the same | as in Fig. 1.
Then GH and G′H′ are also in the same |,
∵ they are || EF, E′F′
And ∵ EF = AB = E′F′, ∴ EE′ = FF′.
For the same reason, GG′ = HH′.
Also AD = BC, AE = BF, AE′ = BF′.

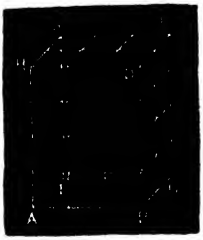
Hence prisms AEE′—DHH′ and BFF′—CGG′ have their bases and faces congruent, and are equal (IV, Theor. 18, Cor. 4). And ∴ taking these equal prisms successively from the whole figure, parallelopiped ABCD—EFGH = parallelopiped ABCD—E′F′G′H′.

Fig. 2.

Secondly, let the edges EF, E'F', be in different | s as in Fig. 2.
Produce H'E', G'F' to meet FE, GH in I, L, J, K.
Then by the first case,

ABCD—EFGH, ABCD—E'F'G'H' each = ABCD—IJKL;
∴ they are equal to one another.

Cor. 1. If three contiguous edges AB, BC, BD, of a rectangular parallelopiped, that is, its length, breadth, and height, contain respectively a, b, c linear units, it contains $a \times b \times c$ cubic units or cubes, each having its edges equal to the linear unit; that is, shortly stated, the volume of a rectangular parallelopiped whose length, breadth, and height are a, b, and c is $= a\,b\,c$.



For if the edges are divided into a, b, and c parts, and planes are drawn through the pts. of division ‖ the faces BCFD, ABDH, and DFGH, the parallelopiped will be divided into small cubes, each cube having a linear unit for its edge; and

the number of cubes = number of cubes in a horizontal layer
× number of layers
= number of squares in DFGH
× number of linear units in BD
$= a \times b \times c$.

NOTE 1. Bearing in mind what is said in the Notes to Theorems 20 and 21 of Book I, it will be seen that a, b, c may be integral or fractional, commensurable or incommensurable.

NOTE 2. This Theorem corresponds to I, Theor. 18.

COR. 2. The volume of any parallelopiped=area of base × altitude.

For any parallelopiped is equal to a parallelopiped of the same altitude on the same base and having its faces adjacent to the base perpendicular to the base; and this parallelopiped being a right prism, is equal to another right prism of the same altitude, on a base of equal area (IV, Th. 18, Cor. 1) which is a rectangle: and the volume of this last mentioned parallelopiped is by the preceding Corollary=area of base × altitude.

Cor. 3. A diagonal plane of a parallelopiped divides it into two triangular prisms of equal volume; and the volume of each $=\frac{1}{2}$ volume of parallelopiped=area of base × altitude.

COR. 4. Hence, since every prism can be divided into triangular prisms by dividing its base into triangles, volume of any prism=area of base × altitude.

THEOREM 20.

Pyramids of equal altitudes and on bases of equal areas are equal.

Fig. 1.

Fig. 2.

Let O—ABCD and *o—a b c* be two pyramids of equal altitudes and on equal bases ABCD, *abc* ; then they are equal.

Divide the altitudes into *n* equal parts, and through the pts. of division draw planes || the bases of the pyramids.

Then the sections made by these planes will be similar and proportional to the bases (IV, Th. 13, 14; III, Th. 8) and as the bases are equal, the sections of the one pyramid will be equal to the corresponding sections of the other.

Now on these sections let prisms be constructed, on the lower side as in fig. 1, and on the upper as in fig. 2, the altitude of each being $\frac{1}{n}$th of that of the pyramid.

Then these prisms in the one pyramid having their bases and altitudes equal to those of the corresponding prisms in the other pyramid, will be respectively equal (IV, Th. 19, Cor. 4).

Let V, v be the volumes of the pyramids,

S, s the sums of the volumes of the prisms,

then s—S=the volume of the lowest prism in *o—abc*.

Now, by increasing n without limit, we can make the height of the lowest pyramid and therefore its volume,

less than any assignable magnitude; so that ultimately, $S=s$.

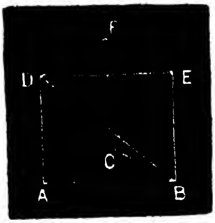
But in that case, the difference between V and S, and that between v and s also vanish,

so that $V=S$, $v=s$.

Hence $V=v$.

COR. 1. The volume of a triangular pyramid D—ABC is one-third of the volume of a prism ABC—DEF having the same base and altitude.

For draw the plane CBD.



Then the pyramids C—ABD and C—EDB which have equal bases ADB, EDB, and a common altitude, namely, the perpendicular from C on plane ABED, are equal.

And the pyramids C—ADB and C—DEF, which have equal bases ABC, DEF, and equal altitudes, namely, the perpendicular distance between planes ABC, DEF, are also equal.

Hence the prism ABC—DEF is divided into three equal pyramids C—ABD, C—EDB, C—DEF; or pyramid C—ADB, that is, D—ABC$=\frac{1}{3}$ of prism ABC—DEF.

COR. 2. The volume of every pyramid is one-third of that of a prism of equal base and altitude.

For it may be divided into triangular pyramids of the same altitude, by dividing the base into triangles, and drawing planes through the dividing lines and the vertex, and then the preceding Corollarly may be applied.

COR. 3. Volume of a pyramid$=\frac{1}{3}\times$area of base$\times$altitude.

X VOLUMES OF CONES, CYLINDERS AND SPHERES.

THEOREM 21.

The volume of a right cone is equal to one-third of the volume of a right cylinder of equal base and altitude.

For the base which is a ⊙ may be divided into
an indefinitely large number of small sectors like O a A,
which may be regarded as triangles,
and right prisms and pyramids constructed upon them,
having the same altitude as the cone,
and the pyramids having the vertex of the cone for their vertex.
Then vol. of each pyramid $= \frac{1}{3}$ of vol. of corresponding prism,
and sum of the vols. of the pyramids $= \frac{1}{3}$ of sum of the vols. of the prisms,
And as these sums are respectively the vols. of the cone and cylinder,
$\therefore$ vol. of the cone $= \frac{1}{3}$ of vol. of cylinder.

Cor. 1. If r=radius of the base, and h=height of the cone, vol. of cylinder $= \pi r^2 h$, vol. of cone $= \frac{1}{3}\pi r^2 h$.

Cor. 2. Area of the convex surface of the cylinder $= 2\pi r h$, (found by dividing the area into elementary rectangles like Aaa' A');

and area of the convex surface of the cone $= \dfrac{2\pi r h'}{2}$,

where h'=slant height AO',
(found by dividing the area into elementary triangles like AaO').

THEOREM 22.

The surface of a sphere is equal to the convex surface of the circumscribed cylinder, and the volume of a sphere is equal to two-thirds of the volume of the circumscribed cylinder.

Let APB be the semi-circle (whose centre is O, and radius$=r$) and ABCD the rectangle (that is, half the circumscribed square) which by their revolution about AB describe the sphere and the cylinder.

Let P, P′ be two pts. on the ⊙ so close to one another that TPP′ may be regarded as a tangent to the ⊙ at P.

Join OP; draw EPN, E′P′N′ ⊥ AB; and let PP′ meet AB in F.

Then $\frac{PP'}{EE'}=\frac{PT}{ET}$ (III, Theor. 1)

$=\frac{OP}{PN}$ (from similar △s EPT, NOP)

$=\frac{EN}{PN}$ (∵ EN = OP);

∴ PN.PP′ = EN. EE′.

Now the area of the surface generated by the chord PP′

= ....curved surface of a frustum of a cone whose vertex is F

$=\frac{1}{3}$ of 2π PN. PF—$\frac{1}{3}$ of 2π P′N′.P′F

$=2\pi\times\frac{1}{3}$ (PN. PF—P′N′.P′F′)

$=2\pi\times\frac{1}{3}\left(PN.\ PF-PN\frac{P'F}{PF}P'F\right)$ $\left(\text{for } \frac{P'N'}{PN}=\frac{P'F}{PF}\right)$

$=2\pi\times\frac{PN}{PF}\times\frac{1}{3}\ (PF^2-P'F^2)$

$=2\pi.\ \frac{PN}{PF}\ \frac{1}{3}\ (PF+P'F)\ (PF-P'F)$

$=2\pi.\ \frac{PN}{PF}\ \frac{1}{3}\ (PF+P'F)\times PP'$

$=2\pi.$ PN. PP′, ultimately, when P, P′ are consecutive pts., that is, when chord PP′ coincides with arc PP′

Hence area of the zone of the sphere generated by PP′
$= 2\pi.\mathrm{PN}.\ \mathrm{PP}' = 2\pi.\ \mathrm{EN}.\ \mathrm{EE}'$ ($\because \mathrm{PN}.\ \mathrm{PP}' = \mathrm{EN}.\ \mathrm{EE}'$)
= area of the surface of the cylinder generated by EE′.
And the whole surface of the sphere
= the whole convex surface of cylinder
$= 2\pi r \times 2r$
$= 4\pi r^2$

To find the volume of the sphere,

we may take any three contiguous pts. on the surface forming a Δ, and suppose the whole surface divided into small Δs like that, and the whole volume divided into small pyramids on these Δs with the centre for their vertex.

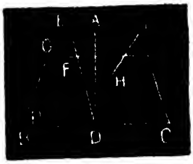
Then volume of each pyramid $= \frac{1}{3}$ of $r \times$ area of base,
and volume of the sphere = sum of the volumes of the pyramids
$= \frac{1}{3} \times r \times$ sum of areas of bases
$= \frac{1}{3} \times r \times$ area of whole surface
$= \frac{1}{3} \times r \times 4\pi r^2$
$= \frac{4}{3}\pi \times r^3$.

SECTION III. PROBLEMS.

I. DRAWING PERPENDICULARS TO PLANES AND STRAIGHT LINES.

PROBLEM 1.

To draw a perpendicular to a plane from a given point without it.



In the given plane, P, take any | BC,
and from the given pt. A draw AD⊥BC.
Then if AD⊥plane P, the thing required is done.
If not, in plane P, draw DE⊥BC, and draw AF⊥DE.
Then AF⊥plane P.
Draw FG ∥ BC.
Then ∵BD⊥AD and ED, ∴BD⊥plane ADE (IV, Th. 4);
and ∵GF∥BD, ∴GF⊥plane ADE (IV, Th. 6),
and ∴GF⊥AF, or AF⊥GF.
Also AF⊥DE;
∴AF⊥plane P which contains DE, GF.

COR. Hence we can draw a perpendicular to the plane P from a given point H in it.

For take any pt. A outside the plane P; draw AF⊥plane P, and from H draw HI∥AF. Then evidently HI⊥plane P.

PROBLEM 2.

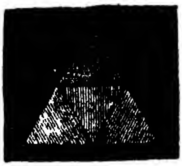
To draw a straight line perpendicular to two straight lines not in the same plane.

Through any pt. B in AB, one of the given | s
draw BE||CD the other given | ;
from any pt. F. in CD draw FG⊥plane ABE;
through G draw GH||CD and meeting AB in H,
and draw HI||GF and meeting, CD in I.
Then HI is the perpendicular required.
For ∵ HI || GF and GF ⊥ plane ABE,
∴ HI ⊥ plane ABE and ∴ ⊥ AB.
Again ∵ GH || CD, and ∠ GHI is a rt. ∠,
∴ ∠ HIF is also a rt. ∠,
that is, HI ⊥ CD.
Hence HI ⊥ AB and CD.

II. Construction of Regular Solids.

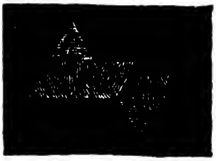
PROBLEM 3.

To construct the five regular solids.

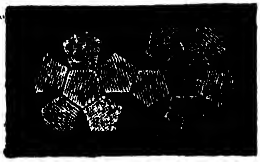


I. Tetrahedron.

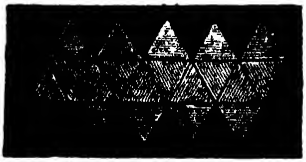
II. Cube.



III. Octahedron.



IV. Dodecahedron.



V. Icosahedron.

Draw on paper, equal equilateral triangles, 4 in number, as in Fig. I, 8 in number, as in Fig. III, 20 in number, as in Fig. V; equal squares, 6 in number, as in Fig. II; and equal regular pentagons, 12 in number, as in Fig. IV.

Cut the paper along the free edges in each Figure, and fold the paper along the joined edges; and five regular solids will be formed, that is, the Tetrahedron from Fig. I, the Cube or Hexahedron, from Fig. II, the Octahedron from Fig. III, the Dodecahedron from Fig IV, and the Icosahedron from Fig. V. And these are the only regular solids (IV, Th. 17, Cor.).

SECTION IV. EXERCISES.

1. The acute angle which a straight line makes with its projection on a plane, is less than the acute angle which it makes with any other straight line meeting it in that plane.

2. If two planes intersect, and straight lines are drawn in one of the planes from any point in their common section, then of these lines, the one drawn perpendicular to the common section has the greatest inclination to the other plane.

3. The angle between two intersecting planes is equal to the angle between their intersecting normals.

4. If a straight line is parallel to each of two intersecting planes, it is parallel to their common section.

5. If a straight line intersects two parallel planes, it makes equal angles with them.

6. If two parallel straight lines intersect the same plane, they make equal angles with it.

7. If three planes intersect one another, their three lines of intersection are either concurrent or parallel.

8. If two planes are drawn, one through each of two parallel straight lines, their common section is parallel to each of those lines.

9. Give the reason why a table with any three legs may stand with all its legs touching a plane floor, but one with four or more legs will not so stand unless the legs are equal or otherwise properly adjusted.

10. There cannot be more than five regular solids. Construct the five regular solids by cutting and folding card-board or stiff paper.

11. Any face angle of a trihedral angle is less than the sum and greater than the difference of the supplements of the other two face angles.

12. The lines joining the centroids of any two faces of a tetrahedron cut each other into segments which are as 1 : 3.

13. Every plane section of a sphere is a circle.

14. Every section of a right cone made by a plane passing through the vertex consists of two intersecting straight lines.

15. Every section of a right cone by a plane perpendicular to the axis is a circle.

16. Shew that the volume of a tank whose top and bottom are rectangles whereof the lengths and breadths are l, l', b, b', whose depth is d, and whose sides have uniform slope, is $\frac{1}{6}d \times \{ \quad + l'b' + (l+l')(b+b')\}$.

(Lilavati, § 221.)

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THE END.

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